Behavior of Visible Submodules in the Class of Multiplication Modules

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Abstract
In this study , we suppose that T is a commutative ring with identity and X is a unitary module on T. A proper submodule W of a module X over a ring T is called visible if for every nonzero ideal I of T , implies $W = IW$ where this concept is up (to our knowledge). Here the behavior of the above concept has been studied within the class of multiplication modules. Some of the distinctive results has been submitted also,the trace of visible submodule has been presented where it was symbolized by $Tr(W)$. Two important descriptions for the trace of visible submodule of multiplication module have been given , also we have demonstrated when the visible submodule of multiplication are torsionless , add to that many properties of $Tr(W)$ have been reviewed.

Keywords

Visible submodule , divisible module , multiplication module , cancellation module , torsionless module , flat module.

Introduction :

T stands for commutative ring with identity and X for the unitary module over T. In [1] , Buthyna N. Shihab and Mahmood S. Fiadh submitted a concept of visible submodule which is defined as a proper submodule W of a module X over a ring T , so that it achieves $W = IW$ for every nonzero ideal I of T. Many of the properties which characterize this concept have been built add to a lot of important results and features have been submitted in [1]. Also , Buthyna N. Shihab and Mahmood S. Fiadh are given in [2] the concept of fully visible module where the module X on T is called fully visible if each submodules of it is visible. The properties and characteristics of this concept have also been reviewed in addition to other results. The aim of this article is to look for the behavior and effectiveness of the visible submodules within the class of multiplication modules. Where many properties have been proved and other important outcomes have been incorporated that adopt the same relationship. In addition to this , the trace of visible submodule has been provided. Two descriptions for the trace of visible submodule which are encoded by $Tr(W)$ have been mentioned. Finally we discussed the conditions under which the trace of visible submodule of multiplication module is torsionless. In our article we will need a number of basic concepts that we will include here.

- An T-module X is called multiplication if $\forall W \leq X$ (W submodule of) , $\exists$ ideal I of Tsuch that $W = IX$ [3].
- Let K , W be two T-submodules of X. Then the residual of K by W is the set of all $s \in T$ such that $sW \sqsubseteq K$ and dented by $(K: W)$. The annihilator of X is written as $(0: X)$ and dented by $ann_T(X)$, if $ann_T(X)$ is equal to zero , then X is said to be faithful [4].
- A submodule W of an T-module X is named multiplication submodule of $X \Leftrightarrow W \cap K = (W: K)K$ for every submodule K of X [5].
- An idempotent submodule W of a module X over T is defined as : W is an idempotent $\Leftrightarrow W = (W: X)W$ [6].
- An T-module X is called cancellation module if $IX = JX$ for any two ideals I and J of , implies $I = J$ [7].
- An T-module X is called fully cancellation module if for each ideal I of T and for each submodules N_1 , N_2 of X such that $IN_1 = IN_2$ implies $N_1 = N_2$ [8].
- An T-module X is flat if for each injective homomorphism $f: N' \longrightarrow N$ from one T-module to another , the homomorphism $I_X \otimes_T f: X \otimes_T N' \longrightarrow X \otimes_T N$ is injective , where I_X is the identity isomorphism of X [6].
- An T-module X is called divisible if and only if $rX = X$ for each $0 \neq r \in T$ [9].

1.Visible submodule of multiplication module

A proper submodule W of an T- module X is said to be visible , if $W = IW$ for every nonzero ideal I of T. In this part , the behavior of visible submodule was studied in the class of multiplication module where a distinction was given to the submodule because we proved that $W \leq X$ is visible if and only if $(W: X)$ is visible ideal of T when X is multiplication faithful finitely generated module. Many properties and useful results have introduced.

Under the class of multiplication and cancellation module , we have the following characterization.

Proposition(1.1):

Let X be a multiplication cancellation T-moduel. Then every proper submoduel N of X is visible submoduel if and only if $(N:X)$ is visible ideal of T.

Proof:

$\Leftarrow$) Suppose that $(N:X)$ is visible ideal of X. Let $x \in N$. Then $(x) \sqsubseteq N$ and hence $((x){:}_{\mathrm{T}} X) \sqsubseteq (N{:}_{\mathrm{T}}X)$.

Therefore $((x){:}_{\mathrm{T}} X) \sqsubseteq (N{:}_{\mathrm{T}}X) = I(N:X)$.

hence $((x){:}_{\mathrm{T}} X)X \sqsubseteq I(N{:}_{\mathrm{T}}X)X$ which implies that $(x) \sqsubseteq IN$

(since X is multiplication module). Therefore $x \in IN$, and hence $\sqsubseteq IN$, also it is clear that $IN \sqsubseteq N$. Thus from two above inclusions , we have $N = IN$, that is N is visible submodule.

$\Longrightarrow$) Let N be a visible submodule , to prove that $(N:X)$ is visible ideal. Let $x \in (N{:}_{\mathrm{T}}X)$. Then $(x)X \sqsubseteq N$, implies $(x)X \sqsubseteq IN$ (since N is visible submodule). Then $(x)X \sqsubseteq I(N:X)X$. But X is cancellation module. Therefore $(x) \sqsubseteq I(N:X)$ and hence $(x) \in I(N:X)$.

Then $(N:X) \sqsubseteq I(N:X)$.

Conversely $I(N:X) \sqsubseteq (N:X)$. Therefore $(N:X) = I(N:X)$. This end the proof.

From proposition (1.1) , we obtain the following corollaries.

Corollary(1.2):

Let N be a proper submodule of a finitely generated faithful multiplication T-module X.Then N is visible if and only if $(N:X)$ is visible ideal of T.

Proof:

Since X is a finitely generated faithful multiplication module , then by ([10], proposition (3-1) , p.52) , we get X is cancellation and by proposition (1.1) we obtain the result.

We can introduce another proof for corollary (1.2) which not depend on proposition (1.1). But at first let us know the following:

A ring T (not necessary commutative) is called (Van Neumann) regular if $\forall t \in T , \exists a \in T$ such that $tat = t$, the purity property has been circulated to the modules by D. Field house [11] , a module X over T is called regular if each submodule W of X is pure in X , that is the inclusion $0 \longrightarrow W \longrightarrow X$ remains exact upon tensoring by any T-module .

Several definition about regular modules were discussed by Ware , Zelamanowitz , and Ramamurthi and Rangaswamy. Anderson and fuller in [9] named the submodule W a pure if $JW = W \cap JX$ for every ideal J of T.

Anther Proof of corollary (1.2):

Let N be a visible submodule of X . Then $N = IN$ for every nonzero ideal $I$of T

But N is pure submodule by ([1] , proposition(2.14)) , therefore $N \cap IX = IN$ for every ideal I of T. Then $N = N \cap IX$ and hence $\big((N \cap IX):X\big) = (N:X)$ which implies $(N:X) \cap (IX:X) = (N:X)$ [12]. Also from [6] , we get $(N{:}_{\mathrm{T}}X)$ is pure ideal of T. Therefore $(N{:}_{\mathrm{T}}X)(IX{:}_{\mathrm{T}}X) = (N:X)$. Thus $(N:X)I = (N:X)$ by [10] , then $I(N:X) = (N:X)$. Thus $(N:X)$ is visible ideal of T.

Conversely :

Suppose that $(N:X)$ is visible ideal of T , then $(N:X) = I(N:X)$ for every nonzero ideal $I$of T. From ([1], proposition (2.14)) we obtain $(N:X)$ is pure ideal of T. Then $I \cap (N:X) = (N:X)$.

which implies $(IX:X) \cap (N:X) = (N:X)$.

And hence $\big((IX \cap N):X\big) = (N:X)$.

But X is multiplication module , then $\big((IX \cap N):X\big)X = (N:X)X$ which implies $IX \cap N = N$, then we get from [6] , N is pure , implies $IN = N$. Thus N is visible submodule

Corollary (1.3):

Let T be an integral domain and X be a faithful cyclic T-module Then N is a visible submodule of X if and only if $(N:X)$ is visible ideal of T.

Proof:

It is known that every cyclic module is multiplication , also every multiplication faithful module over integral domain is finitely generated , by ([10] , proposition (3-3) , p.54) and also by ([10], proposition (3-1) , p.52)) , we have X is cancellation module and by proposition (1.1) , we get the result.

A visible proper ideal J of T is defined as $J = AJ$ for each nonzero ideal A of T [1]. Now we have the following properties.

Proposition(1.4):

Let X be a finitely generated faithful multiplication T-module and I be a proper ideal of T. Then the following are hold:

(1) I is visible ideal of T $\Leftrightarrow$ IX is visible submodule of X.

(2) If N is visible submodule of , then $ann_{\mathrm{T}}(N) = ann_{\mathrm{T}}(N:X)$.

Proof:

(1). $\Longrightarrow$) Let I be a visible ideal of T. Then $JI = I$ for every nonzero ideal J of T and hence $JIX = IX$. Therefore IX is visible submodule.

$\Longleftarrow$) suppose that IX is a visible submodule of X then $JIX = IX$ for all proper ideal J of T (since X is finitely generated faithful multiplication module , then we obtain X is cancellation module by [10]). Therefore $JI = I$ and hence I is visible ideal of T.

(2). Let $x \in ann(N:X)$. Then $x(N:X) = 0$.

Which implies $N = x(N:X)N = 0$, therefore $x \in ann(N)$.

Now , let N be a visible submodule of X. Then $N = IN$ for every nonzero ideal I of T and by ([1] , proposition (2.14)) , we have N is pure , from this fact , we write $N = N \cap IX$ for every ideal I of T. But N is visible , therefore $IN = N \cap IX$. Taking $I = ann_{\mathrm{T}}(N)$ and hence $ann(N)N = N \cap ann(N)$.

$0 = N \cap ann(N)X$.

This leads us $(0:X) = ((N \cap ann(N)X:X)$

$= (N:X) \cap ann(N)X:X)$

$= (N:X) \cap (IX:X)$

$= (N:X) \cap I$

$= (N:X) \cap ann(N)$

$= (N:X)ann(N)$ by ([1] proposition (1.1) and proposition (2.14)).

Then $ann(X) = (N:X)ann(N)$.

But X is faithful which implies that $0 = (N:X)ann(N)$.

Therefore $ann(N) \sqsubseteq ann(N:X)$. Which complets the proof.

The following proposition introduce the necessary conditions for a visible submodule to be multiplication.

Proposition (1.5)

A visible submodule B of a finitely generated faithful multiplication T-module X is multiplication

Proof:

Let A be any submodule of X. Then $A = (A:X)X$ (since X is multiplication module) , we have B is visible submodule of X , then we get $B = IB$ for every a nonzero ideal I of T.

Hence $B \cap A \sqsubseteq B = IB = (A:B)B$ $(choose\ I = (A:B)$.

Therefore $B \cap A \sqsubseteq (A:B)B$ …(1).

Now , it is clear that $(A:B)B \sqsubseteq X$, then $(A:X)(A:B)B \sqsubseteq (A:X)X = A$.

Which implies that $((A:X)(A:B)B) \cap B \sqsubseteq A \cap B$. Hence $(A:B)(A:X)B \sqsubseteq A \cap B$. Now because B is visible submodule of X , then for every nonzero ideal I of T , we have $B = IB$ taking $I = (A:X)$, then

$(A:X)B = B$ and in the last we get $(A:B)B \sqsubseteq A \cap B$…(2).

From(1) and (2) then we obtain $A \cap B = (A:B)B$, that is B is multiplication submodule of X.

The next theorem provide equivalent statements for the visible submodules under certain conditions.

Theorem(1.6):

Suppose X is faithful finitely generated multiplication module over T ,D is proper submodule of X. So all will be equivalent:

1) D is visible submodule of X.
2) D is multiplication and is idempotent in X.
3) D is multiplication and $K = (D{:}X)K$ for each submodule K of D.
4) D is multiplication and $(K{:}D)D = (K{:}X)D$ for each submodule K of X.
5) $\mathrm{T}d = (D{:}X)d$ for each $d \in D$.
6) $\mathrm{T} = (D{:}X) + ann(d)$ for each $d \in D$.

Proof:

(1) $\Longrightarrow$ (2) From proposition (1.5) and ([1] ,proposition (2.18)) .

(2) $\Longrightarrow$ (3) Let D be a proper submodule of X then by (1) , D is multiplication submodule. For each submodule K of D , D is multiplication then $K = (K{:}D)D$.

Also X is multiplication so we will get $K = (K{:}D)(D{:}X)X$. But D is visible submodule , then by corollary (1.2) $(D{:}X)$ is visible ideal. Therefore $(K{:}D)(D{:}X) = (D{:}X)$ and hence $K = (D{:}X)X$.

This leads to

$(K{:}X)K = (K{:}X)(D{:}X)X = (D{:}X)(K{:}X)X = (D{:}X)K$ (since X is multiplication , $K \leq X$).

K is visible because N is visible.

By ([1] , proposition (2.18)) , we have $K = (D{:}X)K$.

Permission (2) $\Longrightarrow$ (3) check.

(3) $\Longrightarrow$ (4) From (3) , we obtain directily D is multiplication. Also , we have D is visible submodule , then $D = ID$ for every nonzero ideal I of T (Taking $I = (K{:}X)$.

Therefore $D = (K{:}X)D$, also we can chose I another ideal of T , that is we can write $= (K{:}D)$, then $(K{:}D)D = D = (K{:}X)D$.

Therefore $(K{:}D)D = (K{:}X)D$.

(4) $\Longrightarrow$ (5) Since D is multiplication , then for every $d \in D$, we have $\mathrm{T}d = (\mathrm{D}{:}X)d$.

(5) $\Longrightarrow$ (6) by (5) , we have , for each $d \in D$ $\exists x \in (\mathrm{D}{:}X)$ $\ni d = xd$. Therefore $D = (x)D$ and hence $(x) = \mathrm{T}$ (since D is cancellation module as a result we get it from the fact that X is faithful FG and multiplication module).

Hence $(\mathrm{D}{:}X) = \mathrm{T}$ which implies $\mathrm{T} + \mathrm{ann_T}(d) = (\mathrm{D}{:}X) + \mathrm{ann_T}(d)$ and hence $\mathrm{T} = (\mathrm{D}{:}X) + \mathrm{ann_T}(d)$. Therefore (6) holds.

(6) $\Longrightarrow$ (1) by (6) , we get $\mathrm{T}D = (D{:}X)D + \mathrm{ann_T}(d)D$ for each $d \in D$.

Therefore $D = (D{:}X)D$.

X is multiplication module , then $D = IX$ for some ideal I of T. Which implies $D = (IX{:}X)D$ (since X is cancellation). Therefore $D = ID$ and hence D is visible submodule of X.

Let us review the following properties.

Proposition (1.7):

Assume X is finitely generated faithful multiplication T-module and K is visible submodule of , then $\bigcap_{k\in I} J_k\, K = (\bigcap_{k\in I} J_k)K$, for every a nonempty collection J_k $(k \in I)$ of visible ideal of T.

Proof:

K is visible submodule of X , then by corollary (1.2)), we have $(K{:}X)$ is visible ideal of T. Suppose that J_k $(k \in I)$ is any collection of visible ideals of T. Now , $(\bigcap_{k\in I} J_k)K = K = (K{:}X)K$ by ([1] , proposition (2.18)) , which is equal

$(K{:}X)(\bigcap_{k\in I} J_k)K = (\bigcap_{k\in I} J_K)(K{:}X)K = (\bigcap_{k\in I} J_k)(K{:}X)AX$ for some ideal A of T. (since X is multiplication module)

we want to show that $(\bigcap_{k\in I} J_k\, K{:}X) = \bigcap_{k\in I} J_k\, (K \dot{:}_{\mathrm{T}} X)$ obviously , $\bigcap_{k\in I} J_k\, (K \dot{:}_{\mathrm{T}} X) \sqsubseteq (\bigcap_{k\in I} J_k\, K{:}X)$. Conversely let ,

$y \in (\bigcap_{k\in I} J_k\, K:X)$. Then $yX \sqsubseteq \bigcap_{k\in I} J_k\, K = \bigcap_{k\in I} J_k\,(K:X)$, but we have X is finitely generated multiplication module , then X is cancellation by [10]. Therefore $y \in \bigcap_{k\in I} J_k\,(K:X)$.

Now , $(\bigcap_{k\in I} J_k)(K:X)AX = (\bigcap_{k\in I} J_k\, K:X)AX$

$= A(\bigcap_{k\in I} J_k\, K:X)X = A(\bigcap_{k\in I} J_k\, K)$

But J_k is visible ideal for all $k \in I$, then by ([1] , corollary(2.9)) we get $\bigcap_{k\in I} J_k$ is visible ideal , also by proposition (1.4) we obtain that $\bigcap_{k\in I} J_k\, K$ is visible , that is $A(\bigcap_{k\in I} I_k\, K) = \bigcap_{k\in I} I_k\, K$ *then* $(\bigcap_{k\in I} J_k\, K) = \bigcap_{k\in I} J_k\, K$ and hence $(\bigcap_{k\in I} J_k)K = \bigcap_{k\in I} J_k\, K$.

<u>Proposition(1.8):</u>

Let X be a multiplication cancellation module over T , and K be a visible submodule of X. Then for each nonzero proper ideal E of T , result from this $E(K \dot{:}_{T} X) = (EK \dot{:}_{T} X)$.

Proof:

K is visible submodule , then $K = EK$ for each nonzero proper ideal E of T , implies $(K \dot{:}_{T} X) = (EK \dot{:}_{T} X)$, also by propositon (1.1), we get $(K \dot{:}_{T} X)$ is visible ideal of T.

Therefore $E(K \dot{:}_{T} X) = (K \dot{:}_{T} X)$ and hence $E(K \dot{:}_{T} X) = (EK \dot{:}_{T} X)$.

After giving above we can demonstrate proof of proposition (1.1) depending on proposition (1.8).

<u>Proof:</u>

$\Longrightarrow$) N is visible submodule of X , then for each a nonzero ideal I of T , we write $N = IN$, therefore $(N:X) = (IN:X)$ and by proposition (1.8) , we obtain $(N:X) = I(N:X)$. Thus we get the result.

$\Longleftarrow$) if $(N:X)$ visible then for each nonzero ideal I of T .

We have $(N:X) = I(N:X)$

And by proposition (1.8) , we obtain $(N:X) = (IN:X)$.

Therefore $(N:X)X = (IN:X)X$ and hence $N = IN$. Thus N is visible submodule.

<u>Proposition (1.9):</u>

Let X be a finitely generated faithful multiplication T-mduel and K be a visible submodule of X. Then $(K:X)$ is the intersection of all visible ideals I of T.

<u>Proof:</u>

Let A be a collection of visible ideals I of T. K is visible submodule of X , then $K = JK$ for every ideal $0 \neq J$ of T , and hence $K = IK$ where $I \in A$, therefore $\bigcap_{I\in A} I$ is visible ideal by ([1] , corollary (2.9)) , this lead us $K = \bigcap_{I\in A} I\,K = (\bigcap_{I\in A} I)K$ by proposition (1.7).

It follows that $(K:X) = \big((\bigcap_{I\in A} I)K:X\big) = (\bigcap_{I\in A} I)(K:X)$, but K is visible , then by proposition (1.1) , $(K:X)$ is visible and hence by ([1] , proposition (2.14)) , $(K:X)$ is pure , therefore

$(K:X) = (\bigcap_{I\in A} I) \cap (K:X)$ and hence $(K:X) \sqsubseteq \bigcap_{I\in A} I$, but K is visible , and hence an idempotent. Therefore $K = (K:X)K$.

It flows that $(K:X) \in A$. So $(K:X) = \bigcap_{I\in A} I$, therefore is the smallest element of A. This ends the proof.

Let's take the next result that shows that each submodule of fully cancellation module be a visible under condition that T is regular ring.

<u>Proposition (1.10):</u>

A proper submodule K of fully cancellation module X over a regular ring T will be visible.

Proof:

T is regular ring , then for every ideal G of T is pure , this leads us to $G^2 = G$ (since every pure ideal is idempotent). Let K be a proper submodule of X then $G^2K = GK$, which implies that $GGK = GK$ note GK , K are two distinct submodules of X and X is fully cancellation , this gives $GK = K$. Therefore K is visible.

Here , we will demonstrate the following results to reach to our important proposition.

<u>Proposition(1.11):</u>

Let T be a PIR and let X be a divisible T -module. Then each proper pure submodule of X is visible.

Proof:

Let I be a nonzero ideal of T and N be a proper pure submodule of X. Since T is PIR , then I = (r) for some $r \in$ T , r $\neq 0$.

We must prove that $N = IN$. It is clearly that $IN \subseteq N$, to prove the another inclusion (that is $N \subseteq IN$) let $n \in N$. Then $n \in N \cap X$. But X is divisible , then $X = rX$ for all $r \in$ T , r $\neq 0$. Therefore $n \in N \cap rX$ which implies that $n \in rN$ (since N is pure submodule).

Therefore $n \subseteq rN$ and hence $n \subseteq IN$, next we obtain $N \subseteq IN$.

Proposition(1.12):

Let X ba a divisible module over a PIR and H be a proper submodule of X. Then the following hold:

(1). If M/H is flat , then H is visible submodule and the converse hold when X is a flat T-module.

(2). If H is a visible submodule of a flat module , then H is flat.

Proof:

(1). From proposition (1.11) and ([13] proposition (2.3),p.20). we get the result of number (1).

(2). From proposition (1.11) and([13], proposition (3.3), p.22). we get the submodule H is flat.

Proposition(1. 13):

Let X be a divisible module over a PIR and H be a proper submodule of X. If X is a multiplication faithful T-module , then H is flat.

Proof:

Since X is multiplication faithful T-module Then $X/H$is also multiplication faithful T-module Then by [14] we obtain that X/H is flat T-module and by the first side of (1) of proposition (1.12) , we get H is visible submodule , and by (2) of proposition (1.12) , we obtain H is flat (since X is multiplication faithful T-module) , then X is flat by [14].

2.Trace of visible submodules

The trace of visible submodule of X over T has been studied here and symbolizes it by $Tr(W)$ and the set $\{\sigma(w): \sigma \in Hom(W, \mathrm{T}), \mathrm{w} \in \mathrm{W}\}$, is a set of generator for $Tr(W)$.

Two important descriptions for the trace of visible submodule of multiplication module have been given , also has been proven when the visible submodules of multiplication modules are torsionless. Where a module Xover T is named torsionless , if X can be embedded in direct product of copies of T [15] , add to that many properties of $Tr(W)$ have been presented.

Proposition (2.1):

If N is a visible submodule of a finitely generated faithful multiplication T-module , then $(N:X) = Tr(N) = \sum_{a \in N} ann(ann(a))$.

Proof:

Suppose that N is visible submodule of , then for each nonzero ideal I of T , we have $N = IN$. Taking $I = (N \dot{:}_{\mathrm{T}} X)$, then $N = (N:X)X$. Therefore $\forall \theta \in Hom(N, \mathrm{T})$, $\theta(N) = \theta((N \dot{:}_{\mathrm{T}} X)N) = (N \dot{:}_{\mathrm{T}} X)\theta(N)$.

Therefore $\sum_{\theta} \theta(N) = (N:X) \sum_{\theta} \theta(N)$ and hence $Tr(N) = (N:X)Tr(N)$, but X is finitely generated faithful multiplication and a submodule N of X is pure by ([1] , proposition (2,14)) , then $(N:X)$ is pure ideal of T , that is

$$Tr(N) = (N:X)Tr(N)$$
$$= (N:X) \cap Tr(N)$$
$$= (N:X) \cap (Tr(N)X:X)$$
$$= ((N \cap Tr(N)X:X)$$
$$= (Tr(N)N:X)$$
$$= (N:X) \text{ (since } N \text{ is visible).}$$

Therefore $Tr(N) = (N:X)$.

Suppose now that $a \in N$ and $\theta \in Hom(N, \mathrm{T})$.

Clearly $ann(a) \sqsubseteq ann(\theta(a))$. Therefore $ann(a)\theta(a) = 0$ and hence $\theta(a) \in ann(ann(a))$.

Which implies $\mathrm{T}\theta(\mathrm{a}) \sqsubseteq \mathrm{ann}(\mathrm{ann}(\mathrm{a})$ and $\theta(N) = \sum_{a\in N} \mathrm{T}\theta(\mathrm{a}) \sum_{a\in N} ann(ann(a))$.

Therefore $Tr(N) \sqsubseteq \mathrm{ann}(\mathrm{ann}(\mathrm{a}))$ and hence $(N:X) \sqsubseteq \mathrm{ann}(\mathrm{ann}(\mathrm{a}))$.

Another side , let $a \in N$.

Then by (theorem (1,6), (6)) we get $\mathrm{T} = (\mathrm{N}:X) + ann(a)$.

Therefore $\mathrm{T}\ \mathrm{ann}(\mathrm{ann}(\mathrm{a})) = (\mathrm{N}:X)ann(ann(a)) + ann(ann(a))ann(a)$ And hence $ann(ann(a)) = (N:X)ann(ann(a)$.

which implies that $ann(ann(a) = Tr(N)ann(ann(a) \sqsubseteq \mathrm{Tr}(\mathrm{N}) = (\mathrm{N}:X)$. Thus $ann(ann(a) \sqsubseteq Tr(N)$. Then we get $Tr(N) = \sum_{a\in N} ann(ann(a))$. This lead us to write $Tr(N) = (\mathrm{N}:X) = \sum_{a\in N} ann(ann(a))$.

Next we review the most important application for proposition (2.1).

Corollary (2.2):

If X is faithful finitely generated multiplication generated module on T and W is visible submodule of , then $Tr(D) = (D \dot{:}_{\mathrm{T}} X) = ann_{\mathrm{T}}(ann_{\mathrm{T}}(D))$.

Proof:

By proposition (2.1) , we achieved the first equality and represented by $Tr(D) = (D \dot{:}_{\mathrm{T}} X)$. The rest is to prove that $(D \dot{:}_{\mathrm{T}} X) = ann_{\mathrm{T}}(ann_{\mathrm{T}}(D))$.

For ease we will write $Ann(D) = ann_{\mathrm{T}}(ann_{\mathrm{T}}(D))$. We want to prove that $Ann(D = (D \dot{:}_{\mathrm{T}} X)$. As D is visible submodule of , then from theorem (2.1) , we have $\mathrm{T} = (D \dot{:}_{\mathrm{T}} X) + ann(N)$.

Therefore $Ann(D) = (D \dot{:}_{\mathrm{T}} X)Ann(D)$ and hence $Ann(D) \subseteq (D \dot{:}_{\mathrm{T}} X)$. Now , let $y \in (D \dot{:}_{\mathrm{T}} X)$. Then $yX \subseteq D$ which implies $y\, ann(N)X = 0$. Therefore $y\, ann(D) \subseteq ann(X) = 0$ (since X is faithful).

Thus $(D \dot{:}_{\mathrm{T}} X) \subseteq Ann(D)$ so that $Ann(D) = (D \dot{:}_{\mathrm{T}} X)$This gives the end of the proof.

Corollary (2.3):

Suppose X is finitely generated faithful multiplication T-module and N is visible submodule of , then

(1) $N = Tr(N)N$.

(2) $ann(N) = ann(Tr(N))$.

Proof:

(1) According to proposition (2.1) , we get $Tr(N) = (N:X)$, then $Tr(N)N = (N:X)N$. But every visible module is an idempotent by ([1] , proposition (2.18)). Therefore $Tr(N)N = N$.

(2) Suppose $r \in ann_{\mathrm{T}}(N)$, then $rN = 0$ implies $\theta_i(rN) = 0$ and hence $\sum_{i=1}^{n} \theta_i(rN) = 0$ this gives $Tr(N) = 0$, so that $r \in ann_{\mathrm{T}}(Tr(N)$ and $ann_{\mathrm{T}}(N) \sqsubseteq ann_{\mathrm{T}}(Tr(N)$.

Let $r \in ann(Tr(N))$.

Then by proposition (2.1) we obtain that $r \in ann_{\mathrm{T}}(N:X)$, and by proposition (1.4), we obtain $r \in ann_{\mathrm{T}}(N)$. This gives the other direction of containment. Thus (2) holds.

Low and smith in [16] Demonstrated that for a multiplication faithful module X if $f \epsilon Hom(X,T)$,$\cap\, ker f = 0$ then X is torsionless.

Corollary (2.4):

Let X be a finitely generated faithful multiplication module over T and D be a visible submodule of X. Then D is torsionless .

Proof:

Suppose that $H = \cap_{\sigma \in Hom(D,\mathrm{T})} Ker\sigma$. Then from proposition (1,5) , D is multiplication , therefore $H = (H:D)D$, follow from this

$0 = \sigma(H) = (H:D)\sigma(D)$ for all $\sigma \in Hom(D, \mathrm{T})$.

Implies that $0 = (H:D)Tr(D)$.

Therefore by proposition (2.1) and corollary (2.3) obtain that $(H:D) \sqsubseteq ann_{\mathrm{T}}(Tr(D) = ann_{\mathrm{T}}([D:X]) = ann_{\mathrm{T}}(D)$.

Finally $H = (H:D)D = 0$. That is the answer.

The coming result of the item offers important properties for the trace of visible module.

Corollary (2.5):

Let X be a faithful finitely generated multiplication module over T and D is visible submodule of X. Let $D = H \oplus L$ for two submodules H and L of X. Then

1) $Tr(D) = Tr(H) \oplus Tr(L)$.

2) $D = Tr(H)D \oplus Tr(L)D$.

Proof:

(1).Since D is visible submodule of X , then H , L are also visible submodules by ([1], proposition (2.7)) and by proposition (2.1) , we obtain

$Tr(D) = (D:X)$, $Tr(H) = (H:X)$, $Tr(L) = (L:X)$. Therefore by [17] and , ([13], proposition (4)) we obtain $Tr(D) = (H + L:X) = (H:X) + (L:X) = Tr(H) + Tr(L)$.

Since X is faithful , then

$0 = (0:_{\mathrm{T}}X) = (H \cap L:X) = (H:X) \cap (L:X) = Tr(H) \cap Tr(L)$.

Hence $Tr(D) = Tr(H) \oplus Tr(L)$.

(2). From corollary (2.3), we have

$D = Tr(D)D$ and by (1) , we get $D = Tr(H)D + Tr(L)D$, also by ([1] , proposition (2.13)) , we have

$Tr(H)D \cap Tr(L)D = \big(Tr(H) \cap Tr(L)\big)D = 0$.

Thus $D = Tr(H)D \oplus Tr(L)D$.

Proposition (2.6):

Let X be a finitely generated faithful multiplication over T , N is visible submodule of X. Then $Tr(N)$ is pure ideal of T.

Proof:

We own N is visible submodule , then for each nonzero ideal A of T , $N = AN$. Thus we take $\theta_i(N) = \theta_i(AN)$.

Therefore $\sum_{i=1}^{n} \theta_i(N) = \sum_{i=1}^{n} \theta_i(AN) = \sum_{i=1}^{n} A\theta_i(N) = A\sum_{i=1}^{n} \theta_i(N)$ and hence $Tr(N) = A\,Tr(N)$.

Now to prove that $Tr(N)$ is pure ideal.

Case (1): $A \cap Tr(N) \sqsubseteq Tr(N) = A\,Tr(N)$ as well as the second case verified and this means $A\,Tr(N) \sqsubseteq A \cap Tr(N)$.

Therefore $A\,Tr(N) = A \cap Tr(N)$. That invites us to say that $Tr(N)$ is pure ideal.

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