Study of a forwarding chain with respect to operators in the Self-maps sub-category

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Abstract: In the following chain we study some backwarding, forwarding and stationary chains in the category Set with respect to some well-known operators like composition, finite product and disjoint union.
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1. Introduction

Our main aim in this text is to study the concept of forwarding (backwarding, stationary) chain in sub-categories of Self-maps in category Set.

In the category C suppose M is a nonempty chain of sub-categories of C (under the inclusion relation, so elements of M are sub-categories of C and for each $\alpha,\beta\in\mathrm{M}$ we have $\alpha\subseteq\beta$ or $\beta\subseteq\alpha$ (since M is a chain)). Also suppose $h:\bigcup\mathrm{M}\to\bigcup\mathrm{M}$ is a map. We say M is [1]:

- a forwarding chain with respect to h if for all $\kappa\in\mathrm{M}$ we have $h(\bigcup\mathrm{M}\setminus\kappa)\subseteq\bigcup\mathrm{M}\setminus\kappa$ (i.e., $h(\bigcup\mathrm{M}\setminus\kappa)\cap\kappa$ is empty),
- a full-forwarding chain with respect to h if it is forwarding and for all distinct $\kappa,\lambda,\mu\in\mathrm{M}$ with $\kappa\subseteq\lambda\subseteq\mu$ there exists $X\in\lambda\setminus\kappa$ with $h(X)\in\mu\setminus\lambda$,
- a backwarding chain with respect to h if for all $\kappa\in\mathrm{M}$ we have $h(\kappa)\subseteq\kappa$,
- a full-backwarding chain with respect to h if it is backwarding and for all distinct $\kappa,\lambda,\mu\in\mathrm{M}$ with $\kappa\subseteq\lambda\subseteq\mu$ there exists $X\in\mu\setminus\lambda$ with $h(X)\in\lambda\setminus\kappa$,
- a stationary chain with respect to h if it is both forwarding and backwarding chain with respect to h.

Let's recall that for equivalence relation E on X and $x\in X$ we have $\frac{x}{E}:=\{y\in X:(x,y)\in E\}$ and quotient space $\frac{X}{E}:=\left\{\frac{z}{E}:z\in X\right\}$. Also $\aleph_0$ denotes the least infinite cardinal number, i.e., $\mathrm{card}(\mathbf{N})=\aleph_0$ (where $\mathbf{N}$ is the collection of all natural numbers).

For self-map $f:X\to X$ consider two equivalence relations $\Im_f$ and $\Re_f$ on X with (see e.g. [2]):

$$(x,y)\in\Im_f \Leftrightarrow f(x)=f(y),$$

$$(x,y)\in\Re_f \Leftrightarrow (\exists n,m\geq 1\ \ f^n(x)=f^m(y)).$$

In this text for cardinal number $\tau>1$ suppose:

- $D_\tau:=\{X\overset{f}{\to}X:$ cardinality of the quotient space $\frac{X}{\Im_f}$ is less than $\tau\}$,
- $E_\tau:=\{X\overset{f}{\to}X:$ cardinality of the quotient space $\frac{X}{\Re_f}$ is less than $\tau\}$.

We denote the sub-category of Set consisting of self-maps by SSet and will denote self-map $f:X\to X$ by (X,f).

2. First operator: *k* times self-composition

In this section consider $k\geq 2$ and $h_1:\mathrm{SSet}\to\mathrm{SSet}$ with $h_1(X,f)=(X,f^k)$ (where $f^k=f\circ\cdots\circ f$ (k times)).

Lemma 1. For $(X,f)\in\mathrm{SSet}$ we have $\Im_f\subseteq\Im_{f^k}$ and $\Re_f=\Re_{f^k}$, thus $\mathrm{card}(\frac{X}{\Im_{f^k}})\leq\mathrm{card}(\frac{X}{\Im_f})$ and $\mathrm{card}(\frac{X}{\Re_{f^k}})=\mathrm{card}(\frac{X}{\Re_f})$.

Proof. For each $(X,f)\in\mathrm{SSet}$ and $(x,y)\in\Im_f$ we have $f(x)=f(y)$ thus $f^k(x)=f^k(y)$ and $(x,y)\in\Im_{f^k}$, therefore $\Im_f\subseteq\Im_{f^k}$ and

$$\begin{array}{c}\frac{X}{\Im_f}\to\frac{X}{\Im_{f^k}}\\ \frac{z}{\Im_f}\mapsto\frac{z}{\Im_{f^k}}\end{array}$$

is onto, hence $\mathrm{card}(\frac{X}{\Im_{f^k}})\leq\mathrm{card}(\frac{X}{\Im_f})$. Moreover, $x,y\in X$ we have:

$(x,y)\in\Re_f\Leftrightarrow\exists n,m\geq 1(f^m(x)=f^n(y))$

$$\Leftrightarrow \exists n,m \geq 1\,(f^{mk}(x) = f^{nk}(y))$$
$$\Leftrightarrow \exists n,m \geq 1\,((f^{k})^{m}(x) = (f^{k})^{n}(y))$$
$$\Leftrightarrow (x,y) \in \Re_{f^k}.$$

Which leads to $\Re_f = \Re_{f^k}$ and completes the proof.

Theorem 2. Consider nonempty sub-class M of $\{\text{SSet}\} \cup \{D_\tau : \tau > 1\}$:

a. M is backwarding with respect to h_1.

b. M is forwarding (resp. stationary) with respect to h_1 iff M is singleton,

Proof. (a) By Lemma 1, $h_1(D_\tau) \subseteq D_\tau$ for each $\tau > 1$, thus $h_1(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$ and M is backwarding with respect to h_1.

(b) Now suppose M has at least two elements and consider distinct elements $H, K \in \mathrm{M}$ with $H \subset K$. There exists $\tau > 1$ with $H = D_\tau$. Choose cardinal number $\theta > 0$ with $\tau = \theta + 1$. Consider arbitrary set A with $\text{card}(A) = \theta$ and $b \notin A \times \{0,1\}$ (e.g., $b = (0,-1)$). Let $X = (A \times \{0,1\}) \cup \{b\}$ and define $f : X \to X$ with $f(a,0) = (a,1)$, $f(a,1) = b$ and $f(b) = b$. Then

$$\frac{X}{\Im_f} = \{\{(a,0)\} : a \in A\} \cup \{(A \times \{1\}) \cup \{b\}\}$$

and $\text{card}(\frac{X}{\Im_f}) = \theta + 1 = \tau$. Thus $(X,f) \notin D_\tau = H$ and for each $\psi > \tau$ we have $(X,f) \in D_\psi \subset \text{SSet}$, in particular $(X,f) \in K \setminus C \subseteq \bigcup \mathrm{M} \setminus C$. On the other hand $\frac{X}{\Im_{f^k}} = \{X\}$, hence $h_1(X,f) = (X, f^k) \in D_2 \subseteq D_\tau = C$.

Therefore, M is not forwarding (resp. stationary) with respect to h_1.

Corollary 3. Each nonempty sub-class M of $\{\text{SSet}\} \cup \{E_\tau : \tau > 1\}$, is stationary (resp. forwarding, backwarding) with respect to h_1.

Proof. Use Lemma 1.

3. Second operator: finite *k* times self-product

For $k \geq 2$ consider $h_2 : \text{SSet} \to \text{SSet}$ with $h_2(X,f) = (X^k, f_k)$, $f_k(y_1,\cdots,y_k) = (f(y_1),\cdots,f(y_k))$.

Lemma 4. Consider $(X,f) \in \text{SSet}$:

1. we have:

$$\text{card}(\frac{X^k}{\Im_{f_k}}) = \left(\text{card}(\frac{X}{\Im_f})\right)^k \begin{cases} < \aleph_0 & \dfrac{X}{\Im_f} \text{ is finite} \\ = \text{card}(\dfrac{X}{\Im_f}) & \text{otherwise} \end{cases}.$$

In particular for $\tau \in \{\theta : \theta = 2 \vee \theta \geq \aleph_0\}$, $(X,f) \in D_\tau$ iff $h_2(X,f) \in D_\tau$.

2. we have:

$$\text{card}(\frac{X}{\Re_f}) \leq \text{card}(\frac{X^k}{\Re_{f_k}}) \leq \left(\text{card}(\frac{X}{\Re_f})\right)^k.$$

In particular for $\tau \in \{\theta : \theta = 2 \vee \theta \geq \aleph_0\}$, $(X,f) \in E_\tau$ iff $h_2(X,f) \in E_\tau$.

Proof. (1) For $(x_1,\cdots,x_k),(y_1,\cdots,y_k) \in X^k$ we have:

$$((x_1,\cdots,x_k),(y_1,\cdots,y_k)) \in \Im_{f_k}$$
$$\Leftrightarrow f_k(x_1,\cdots,x_k) = f_k(y_1,\cdots,y_k)$$
$$\Leftrightarrow (f(x_1),\cdots,f(x_k)) = (f(y_1),\cdots,f(y_k))$$
$$\Leftrightarrow (x_1,y_1),\ldots,(x_k,y_k) \in \Im_f$$

so

$$\begin{array}{c} \dfrac{X^k}{\Im_{f_k}} \to \left(\dfrac{X}{\Im_f}\right)^k \\ \frac{(z_1,\cdots,z_k)}{\Im_{f_k}} \mapsto (\frac{z_1}{\Im_f},\cdots,\frac{z_k}{\Im_f}) \end{array}$$

is bijective and $\text{card}(\frac{X^k}{\Im_{f_k}}) = \left(\text{card}(\frac{X}{\Im_f})\right)^k$.

(2) For $((x_1,\cdots,x_k),(y_1,\cdots,y_k)) \in \Re_{f_k}$ there exist $n,m \geq 1$ with $f_k^n(x_1,\cdots,x_k) = f_k^m(y_1,\cdots,y_k)$ thus for all $i \in \{1,\ldots,k\}$ we have $f^n(x_i) = f^m(y_i)$ and $(x_i,y_i) \in \Re_f$ so

$$\begin{array}{c} \dfrac{X^k}{\Re_{f_k}} \to \left(\dfrac{X}{\Re_f}\right)^k \\ \frac{(z_1,\cdots,z_k)}{\Re_{f_k}} \mapsto (\frac{z_1}{\Re_f},\cdots,\frac{z_k}{\Re_f}) \end{array}$$

is onto, thus $\text{card}(\frac{X^k}{\Re_{f_k}}) \leq \left(\text{card}(\frac{X}{\Re_f})\right)^k$, moreover

$$\begin{array}{c} \dfrac{X}{\Re_f} \to \dfrac{X^k}{\Re_{f_k}} \\ \frac{z}{\Re_f} \mapsto \frac{(z,\cdots,z)}{\Re_{f_k}} \end{array}$$

is one-to-one, hence $\text{card}(\frac{X}{\Re_f}) \leq \text{card}(\frac{X^k}{\Re_{f_k}})$.

Theorem 5. Consider nonempty sub-class M of $\{\text{SSet}\} \cup \{D_\tau : \tau > 1\}$ we have:

1. The following statements are equivalent:
   a. $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$,
   b. one of the following conditions occurs:
   - $\mathrm{M} \cap (\{\text{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\})$ is nonvoid,
   - $\mathrm{M} \cap \{D_\tau : \tau < \aleph_0\}$ is infinite,
   - $\mathrm{M} = \{D_2\}$,

   c. $D_{\aleph_0} \subseteq \bigcup \mathrm{M}$ or $\mathrm{M} = \{D_2\}$,
2. M is forwarding with respect to h_2 iff $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$,
3. M is backwarding (resp. stationary) with respect to h_2 iff $\mathrm{M} \subseteq \{\text{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\} \cup \{D_2\}$.

Proof. (1) (a) $\Rightarrow$ (b): Suppose $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$, $\mathrm{M} \cap (\{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\})$ is empty, and $\mathrm{M} \cap \{D_\tau : \tau < \aleph_0\}$ is finite, then there exist $n_1 < \cdots < n_s = p < \aleph_0$ with $\mathrm{M} = \{D_{n_j} : 1 \leq j \leq s\}$. So $\bigcup \mathrm{M} = D_p$, if $p > 2$ then consider $X = \{1, \ldots, p\}$ and $f : X \to X$ with $f(i) = i+1$ for $i < p$ and $f(p) = p$, therefore

$$\frac{X}{\mathfrak{I}_f} = \{\{i\} : 1 \leq i \leq p-2\} \cup \{\{p-1, p\}\},$$

$\mathrm{card}(\frac{X}{\mathfrak{I}_f}) = p-1 < p$ and $(X, f) \in D_p$. By Lemma 4(1),

$$\mathrm{card}(\frac{X^k}{\mathfrak{I}_{f_k}}) = (p-1)^k \geq 2(p-1) > p$$

and $h_2(X, f) = (X^k, f_k) \notin D_p = \bigcup \mathrm{M}$ which is in contradiction with $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$. Hence $n_s = 2$ and $\mathrm{M} = \{D_2\}$.

(b) $\Rightarrow$ (c): It's clear by definition of D_τ s.

(c) $\Rightarrow$ (a): Since for each transfinite cardinal number τ we have $\tau^k = \tau$ by Lemma 4(1) for each transfinite cardinal number τ we have $h_2(D_\tau) \subseteq D_{\tau^k} = D_\tau$ also for each $2 < n < \aleph_0$ we have $h_2(D_n) \subseteq D_{(n-1)^k+1} \subseteq D_{\aleph_0}$ moreover $h_2(D_2) \subseteq D_2$ which leads to the desired result.

(2) Use (1) and Lemma 4(1).

(3) First suppose $\mathrm{M} \subseteq \{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\} \cup \{D_2\}$, then by item (1), $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$. Using Lemma 4(1), M is backwarding and stationary with respect to h_2.

Now suppose M is backwarding with respect to h_2 and $\mathrm{M} \not\subseteq \{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\} \cup \{D_2\}$. Then there exists finite $p > 2$ with $D_p \in \mathrm{M}$. Using the same method described in the proof of "(a) $\Rightarrow$ (b)" in item (1), there exists $(X, f) \in D_p$ with $h_2(X, f) \notin D_p$, which is a contradiction and completes the proof.

Theorem 6. Consider nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau > 1\}$ we have:

1. The following statements are equivalent:
   a. $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$,
   b. one of the following conditions occurs:
   - $\mathrm{M} \cap (\{\mathrm{SSet}\} \cup \{E_\tau : \tau \geq \aleph_0\})$ is nonvoid,
   - $\mathrm{M} \cap \{E_\tau : \tau < \aleph_0\}$ is infinite,
   - $\mathrm{M} = \{E_2\}$,

   c. $E_{\aleph_0} \subseteq \bigcup \mathrm{M}$ or $\mathrm{M} = \{E_2\}$,
2. M is forwarding with respect to h_2 iff $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$,
3. M is backwarding (resp. stationary) with respect to h_2 iff $\mathrm{M} \subseteq \{\mathrm{SSet}\} \cup \{E_\tau : \tau \geq \aleph_0\} \cup \{E_2\}$.

Proof. For finite $p > 2$ consider $X = \{1, \ldots, p-1\}$ and identity map $f : X \underset{x \mapsto x}{\to} X$, then $\mathrm{card}(\frac{X}{\mathfrak{R}_f}) = p-1$ and $(X, f) \in E_p$. However, $\mathrm{card}(\frac{X^k}{\mathfrak{R}_{f_k}}) = (p-1)^k \geq p$ and $h_2(X, f) \notin E_p$. Use Lemma 4(2) and a similar method described in the proof of Theorem 5 to complete the proof.

Note 7 (infinite self-product). For arbitrary infinite set Γ consider $h : \mathrm{SSet} \to \mathrm{SSet}$ with $h(X, f) = (X^\Gamma, f_\Gamma)$ with $f_\Gamma((x_i)_{i\in\Gamma}) = (f(x_i))_{i\in\Gamma}$. Then using similar method described in the finite case for each $(X, f) \in \mathrm{SSet}$ we have

$$\mathrm{card}(\frac{X^\Gamma}{\mathfrak{I}_{f_\Gamma}}) = \left(\mathrm{card}(\frac{X}{\mathfrak{I}_f})\right)^{\mathrm{card}(\Gamma)}$$

and

$$\mathrm{card}(\frac{X}{\mathfrak{R}_f}) \leq \mathrm{card}(\frac{X^\Gamma}{\mathfrak{R}_{f_\Gamma}}) \leq \left(\mathrm{card}(\frac{X}{\mathfrak{R}_f})\right)^{\mathrm{card}(\Gamma)}.$$

Thus for any nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{D_\tau : \tau > 1\}$ with $\mathrm{SSet} \in \mathrm{M}$, M is forwarding with respect to h. Also for nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq 2^{\mathrm{card}(\Gamma)}\}$, M is stationary with respect to h. Also for any nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau > 1\}$ with $\mathrm{SSet} \in \mathrm{M}$, M is forwarding with respect to h. Also for nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau \geq 2^{\mathrm{card}(\Gamma)}\}$, M is stationary with respect to h.

4. Third operator: disjoint union

Consider arbitrary set Γ with at least two elements and $h_3 : \mathrm{SSet} \to \mathrm{SSet}$ where $h_3(X, f) = (X \times \Gamma, f_{(\Gamma)})$ and $f_{(\Gamma)}(x, \gamma) = (f(x), \gamma)$ (as a matter of fact one may consider $h_3(X, f)$ "looks like" Γ copies disjoint union of (X, f)).

Lemma 8. For each $(X, f) \in \mathrm{SSet}$ we have:

$$\mathrm{card}(\frac{X \times \Gamma}{\mathfrak{I}_{f_{(\Gamma)}}}) = \mathrm{card}(\Gamma)\,\mathrm{card}(\frac{X}{\mathfrak{I}_f})$$

and

$$\mathrm{card}(\frac{X \times \Gamma}{\mathfrak{R}_{f_{(\Gamma)}}}) = \mathrm{card}(\Gamma)\,\mathrm{card}(\frac{X}{\mathfrak{R}_f}).$$

Proof. For each $(X, f) \in \mathrm{SSet}$ and $(x, i), (y, j) \in X \times \Gamma$ we have:

$$((x,i),(y,j)) \in \mathfrak{I}_{f_{(\Gamma)}} \Leftrightarrow (x, y) \in \mathfrak{I}_f \wedge i = j$$

and

$$((x,i),(y,j)) \in \mathfrak{R}_{f_{(\Gamma)}} \Leftrightarrow (x, y) \in \mathfrak{R}_f \wedge i = j.$$

Thus:

$$\underset{\frac{(z,\gamma)}{\Im_{f(\Gamma)}}\mapsto(\frac{z}{\Im_f},\gamma)}{\frac{X\times\Gamma}{\Im_{f(\Gamma)}}\to\frac{X}{\Im_f}\times\Gamma}$$

and

$$\underset{\frac{(z,\gamma)}{\Im_{f(\Gamma)}}\mapsto(\frac{z}{\Im_f},\gamma)}{\frac{X\times\Gamma}{\Re_{f(\Gamma)}}\to\frac{X}{\Re_f}\times\Gamma}$$

are bijective which lead to the desired result.

Theorem 9 (finite disjoint union). For finite Γ (with at least two elements) consider nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>1\}$ and nonempty sub-class M' of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau>1\}$, we have:

1. The following statements are equivalent:
   a. $h_3(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$,
   b. one of the following conditions occurs:
   - $\mathrm{M}\cap(\{\mathrm{SSet}\}\cup\{D_\tau:\tau\geq\aleph_0\})$ is nonvoid,
   - $\mathrm{M}\cap\{D_\tau:\tau<\aleph_0\}$ is infinite,

   c. $D_{\aleph_0}\subseteq\bigcup\mathrm{M}$,
2. M is forwarding with respect to h_3 iff $h_3(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$,
3. M is backwarding (resp. stationary) with respect to h_3 iff $\mathrm{M}\subseteq\{\mathrm{SSet}\}\cup\{D_\tau:\tau\geq\aleph_0\}$,
4. The following statements are equivalent:
   a. $h_3(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$,
   b. one of the following conditions occurs:
   - $\mathrm{M}'\cap(\{\mathrm{SSet}\}\cup\{E_\tau:\tau\geq\aleph_0\})$ is nonvoid,
   - $\mathrm{M}'\cap\{E_\tau:\tau<\aleph_0\}$ is infinite,

   c. $E_{\aleph_0}\subseteq\bigcup\mathrm{M}'$,
5. M' is forwarding with respect to h_3 iff $h_3(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$,
6. M' is backwarding (resp. stationary) with respect to h_3 iff $\mathrm{M}\subseteq\{\mathrm{SSet}\}\cup\{E_\tau:\tau\geq\aleph_0\}$.

Proof. For finite $p>1$ consider $X=\{1,\ldots,p-1\}$ and identity map $f:X\underset{x\mapsto x}{\to}X$ as in the proof of Theorem 6, then $\mathrm{card}(\frac{X}{\Im_f})=\mathrm{card}(\frac{X}{\Re_f})=p-1$ and $(X,f)\in E_p\cap D_p$. However

$$\mathrm{card}(\frac{X\times\Gamma}{\Im_{f(\Gamma)}})=\mathrm{card}(\frac{X\times\Gamma}{\Re_{f(\Gamma)}})=(p-1)\,\mathrm{card}(\Gamma)>p$$

and $h_3(X,f)\notin D_p\cup E_p$. Use Lemma 8 and a similar method described in Theorems 5 and 6 to complete the proof.

Note 10 (infinite disjoint union). For infinite Γ and nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>1\}$ with $\mathrm{SSet}\in\mathrm{M}$, M is forwarding with respect to h_3. Also for nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>\mathrm{card}(\Gamma)\}$, M is stationary with respect to h_3. Also for any nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau>1\}$ with $\mathrm{SSet}\in\mathrm{M}$, M is forwarding with respect to h_3. Also for nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau\geq\mathrm{card}(\Gamma)\}$, M is stationary with respect to h_3.

5. Fourth operator: induced map on power set

For arbitrary set X and cardinal numbers ϑ,θ let

$$\mathrm{P}^{<\vartheta}_{>\theta}(X)=\{A\subseteq X:\theta<\mathrm{card}(A)<\vartheta\}$$

$$\mathrm{P}^{<\vartheta}(X)=\{A\subseteq X:\mathrm{card}(A)<\vartheta\},$$

and $h_4:\mathrm{SSet}\to\mathrm{SSet}$ with $h_4(X,f)=(\mathrm{P}(X),\mathrm{P}(f))\in\mathrm{SSet}$ where $\mathrm{P}(f)(A)=f(A)(=\{f(x):x\in A\})$ (for $A\subseteq X$) also $h_4^{<\vartheta}(X,f)=(\mathrm{P}^{<\vartheta}(X),\mathrm{P}^{<\vartheta}(f))\in\mathrm{SSet}$ as the restriction of the above self-map to $\mathrm{P}^{<\vartheta}(X)$, i.e. $\mathrm{P}^{<\vartheta}(f)=\mathrm{P}(f)|_{\mathrm{P}^{<\vartheta}(X)}$.

Lemma 11. For $1<k<\aleph_0$ we have:

$$\mathrm{card}(\frac{X}{\Im_f})\leq\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})\leq\left(\mathrm{card}(\frac{X}{\Im_f})\right)^{2k-1}+1.$$

In particular for infinite $\frac{X}{\Im_f}$ we have $\mathrm{card}(\frac{X}{\Im_f})=\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})$.

Proof. For each nonempty $A,B\in\mathrm{P}^{<k+1}(X)$ (i.e. $A,B\in\mathrm{P}^{<k+1}_{>0}(X)$) there exist $x_1,\ldots,x_k,y_1,\ldots,y_k\in X$ (may be not distinct) with $A=\{x_1,\ldots,x_k\},B=\{y_1,\ldots,y_k\}$. Now for $(X,f)\in\mathrm{SSet}$ and nonempty $A,B\in\mathrm{P}^{<k+1}(X)$ with $(A,B)\in\Im_{\mathrm{P}^{<k+1}(f)}$ and $A=\{x_1,\ldots,x_k\},B=\{y_1,\ldots,y_k\}$, we have

$$\mathrm{P}^{<k+1}(f)(A)=\mathrm{P}^{<k+1}(f)(B)$$

thus $\{f(x_1),\ldots,f(x_k)\}=\{f(y_1),\ldots,f(y_k)\}$, so for each $i\in\{1,\ldots,k\}$ there exist $s_i,t_i\in\{1,\ldots,k\}$ with $f(x_i)=f(y_{s_i})$ and $f(y_i)=f(x_{t_i})$. Without any loss of generality we may assume $f(x_1)=f(y_1)$ and $s_1=t_1=1$. Thus

$$(f(x_1),\cdots,f(x_k),f(x_{t_2}),\cdots,f(x_{t_k}))$$
$$=(f(y_{s_1}),\cdots,f(y_{s_k}),f(y_2),\cdots,f(y_k))$$

using the same notations as in the Second study we have $f_{2k-1}(x_1,\cdots,x_k,x_{t_2},\cdots,x_{t_k})=f_{2k-1}(y_{s_1},\cdots,y_{s_k},y_2,\cdots,y_k)$ and $((x_1,\cdots,x_k,x_{t_2},\cdots,x_{t_k}),(y_{s_1},\cdots,y_{s_k},y_2,\cdots,y_k))\in\Im_{f_{2k-1}}$ moreover, clearly we have

$$\{x_1,\cdots,x_k,x_{t_2},\cdots,x_{t_k}\}=\{x_1,\cdots,x_k\}$$

and $\{y_{s_1},\cdots,y_{s_k},y_2,\cdots,y_k\}=\{y_1,\cdots,y_k\}$. Hence the following map is onto

$$\begin{array}{c}\left\{\frac{(z_i)_{1\leq i\leq 2k-1}}{\Im_{f_{2k-1}}}:\mathrm{card}\{z_1,\ldots,z_{2k-1}\}\leq k\right\}\to\frac{\mathrm{P}_{>0}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}\\ \frac{(z_i)_{1\leq i\leq 2k-1}}{\Im_{f_{2k-1}}}\mapsto\frac{\{z_1,\ldots,z_{2k-1}\}}{\Im_{\mathrm{P}^{<k+1}(f)}}\end{array}$$

(by $\frac{\mathrm{P}_{>0}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}$ we mean $\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}$ except the equivalence class of empty set).

Therefore (use the Section 3 too):

$$\mathrm{card}(\frac{\mathrm{P}_{>0}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})\leq\mathrm{card}(\frac{X^{2k-1}}{\Im_{f_{2k-1}}})=\left(\mathrm{card}(\frac{X}{\Im_f})\right)^{2k-1}.$$

hence:

$$\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})\leq\mathrm{card}(\frac{X^{2k-1}}{\Im_{f_{2k-1}}})+1=\left(\mathrm{card}(\frac{X}{\Im_f})\right)^{2k-1}+1$$

Moreover:

$$\begin{array}{c}\frac{X}{\Im_f}\to\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}\\ \frac{z}{\Im_f}\mapsto\frac{\{z\}}{\Im_{\mathrm{P}^{<k+1}(f)}}\end{array}$$

is one-to-one, thus

$$\mathrm{card}(\frac{X}{\Im_f})\leq\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})\leq\left(\mathrm{card}(\frac{X}{\Im_f})\right)^{2k-1}+1.$$

Corollary 12. For $1<k<\aleph_0$ we have:

$$\mathrm{card}(\frac{X}{\Re_f})\leq\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Re_{\mathrm{P}^{<k+1}(f)}})\leq\left(\mathrm{card}(\frac{X}{\Re_f})\right)^{2k-1}+1.$$

Proof. Use a similar method described in Lemma 11.

Note 13. For $1<k<\aleph_0$, finite Γ (with at least two elements) nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>1\}$ and nonempty sub-class M' of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau>1\}$, we have:

- $h_3(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$ iff $h_4^{<k+1}(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$,
- M is forwarding (respectively backwarding, stationary) with respect to $h_4^{<k+1}$ iff it is forwarding (respectively backwarding, stationary) with respect to h_3,
- $h_3(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$ iff $h_4^{<k+1}(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$,
- M' is forwarding (respectively backwarding, stationary) with respect to $h_4^{<k+1}$ iff it is forwarding (respectively backwarding, stationary) with respect to h_3.

Proof. Use Theorem 9.

References

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