Set-theoretical entropy of Alexandroff square homeomorphisms

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Abstract: In the following text for Alexandroff square $\mathbf{A}$,and unit square $\mathbf{O}$ (also equal to $[0,1]\times[0,1]$) equipped with lexicographic order topology if $X\in\{\mathbf{A},\mathbf{O}\}$ for homeomorphism $f:X\to X$ we have $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ moreover $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if f^4 is the identity map on X (where $\mathrm{ent}_{\mathrm{set}}(f)$ denotes set-theoretical entropy of f).

Keywords: Alexandroff square, lexicographic order, set-theoretical entropy.

1. Introduction

Several topologies have been introduced on unit square $[0,1]\times[0,1]$, like induced Euclidean topology, lexicographic order topology, Alexandroff square, etc.. In this text we consider $\mathbf{A}:=[0,1]\times[0,1]$ under topology generated by basis consisting of [3]:

- $\{t\}\times(U\setminus\{t\})$ where $t\in[0,1]$ and U is an open subset of $[0,1]$ (as a subset of real line $\mathbf{R}$),
- $([0,1]\setminus F)\times U$ where F is a finite subset of $[0,1]$ and U is an open subset of $[0,1]$ (as a subset of real line $\mathbf{R}$).
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On the other hand several entropies have been introduced, e.g., topological entropy, algebraic entropy, adjoint entropy, set-theoretical entropy, etc.. Here we deal with set-theoretical entropy which has been introduced for the first time in [1]. For arbitrary set D, self-map $\lambda:D\to D$ and finite subset B of D the limit $h(B,\lambda):=\lim\limits_{n\to\infty}\frac{|B\cup\lambda(B)\cup\cdots\cup\lambda^{n-1}(B)|}{n}$ exists (where $|K|$ denotes the cardinality of finite set K). Define set-theoretical entropy of $\lambda:D\to D$ as $\sup\{h(F,\lambda):F$ is a finite subset of $D\}$ and denote it with $\mathrm{ent}_{\mathrm{set}}(\lambda)$.

In this text we compute all possible set-theoretical entropies of homeomorphism on Alexandroff square $\mathbf{A}$.

Remark 1.1. For $\lambda:D\to D$, $\mathrm{ent}_{\mathrm{set}}(\lambda)=\sup(\{n:$ there exist $x_1,\ldots,x_n\in D$ such that $\{\lambda^k(x_1)\}_{k\geq1},\ldots,\{\lambda^k(x_n)\}_{k\geq1}$ are n pairwise disjoint one-to-one sequences $\}\cup\{0\})$ [1]. Moreover for $t\geq1$ we have $\mathrm{ent}_{\mathrm{set}}(\lambda^t)=t\,\mathrm{ent}_{\mathrm{set}}(\lambda)$.

Convention 1.2. Using the same notations as in [2], by $<x,y>$ we mean ordered set $\{x,\{x,y\}\}$, and by (a,b) we mean open interval $\{z\in\mathbf{R}:a<z<b\}$, also in set $[0,1]\times[0,1]$, let $\Delta:=\{<t,t>:t\in[0,1]\}$ and:

$$\mathrm{P}_1:=<0,0>,\mathrm{P}_2:=<0,1>,\mathrm{P}_3:=<1,1>,\mathrm{P}_4:=<1,0>,$$
$$\mathrm{L}_1:=\{0\}\times(0,1),\mathrm{L}_2:=(0,1)\times\{1\},$$
$$\mathrm{L}_3:=\{1\}\times(0,1),\mathrm{L}_4:=(0,1)\times\{0\}.$$

2. Set-theoretical entropy of homeomorphisms of A

Lemma 2.1. For order preserving bijection $f:[0,1]\to[0,1]$ the following statements are equivalent:

- $\mathrm{ent}_{\mathrm{set}}(f)>0$,
- $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$,
- $f\neq\mathrm{id}_{[0,1]}$,

i.e., $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if $f=\mathrm{id}_{[0,1]}$.

Proof. Suppose $f\neq\mathrm{id}_{[0,1]}$, then there exists $t\in[0,1]$ with $f(t)\neq t$, without any loss of generality we may suppose $t<f(t)$ for $n\geq1$ choose $t=x_1<x_2<\cdots<x_n<f(t)$, then $t=x_1<x_2<\cdots<x_n<f(t)=f(x_1)<f(x_2)<\cdots<f(x_n)<f^2(x_1)<f^2(x_2)<\cdots<f^2(x_n)<\cdots$ and the sequences $\{f^k(x_1)\}_{k\geq1},\ldots,\{f^k(x_n)\}_{k\geq1}$ are pairwise disjoint and one-to-one, so by Remark 1.1 we have $\mathrm{ent}_{\mathrm{set}}(f)\geq n$. Hence $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$.

Remark 2.2. In Alexandroff square $\mathbf{A}$, for homeomorphism $f:\mathbf{A}\to\mathbf{A}$ we have $f(\Delta)=\Delta$ also for all $t\in[0,1]$ there exists $s\in[0,1]$ such that $f(\{t\}\times[0,1])=\{s\}\times[0,1]$ in addition $g:[0,1]\to[0,1]$ with $f<t,x>=<s,g(x)>$ is a homeomorphism. Moreover exactly one of the following conditions occurs [2]:

- $f(\mathrm{P}_i)=\mathrm{P}_i(i=1,2,3,4)$, $f(\mathrm{L}_1)=\mathrm{L}_1$, $f(\mathrm{L}_3)=\mathrm{L}_3$,
- $f(\mathrm{P}_1)=\mathrm{P}_3$, $f(\mathrm{P}_2)=\mathrm{P}_4$, $f(\mathrm{P}_3)=\mathrm{P}_1$, $f(\mathrm{P}_4)=\mathrm{P}_2$, $f(\mathrm{L}_1)=\mathrm{L}_3$, $f(\mathrm{L}_3)=\mathrm{L}_1$.

Theorem 2.3. In Alexandroff square $\mathbf{A}$, for homeomorphism $f:\mathbf{A}\to\mathbf{A}$ the following statements are equivalent:

- $\mathrm{ent}_{\mathrm{set}}(f)>0$,
- $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$,
- $f^4\neq \mathrm{id}_{\mathbf{A}}$,

i.e., $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if $f^4=\mathrm{id}_{\mathbf{A}}$.

Proof. Suppose $\mathrm{ent}_{\mathrm{set}}(f)>0$. By Remark 1.1, we have $\mathrm{ent}_{\mathrm{set}}(f^2)>0$. Moreover considering homeomorphism $f^2:\mathbf{A}\to\mathbf{A}$ by Remark 2.2 we have $f^2(\mathrm{P}_i)=\mathrm{P}_i$ $(i=1,2,3,4)$, also $f^2|_\Delta:\Delta\to\Delta$ is a homeomorphism. Note that Δ as a subspace of $\mathbf{A}$ has the same topology as a subspace of plane $\mathbf{R}^2$. Considering homeomorphism $h:[0,1]\to\Delta$ with $h(t)=<t,t>(t\in[0,1])$, we have homeomorphism $h^{-1}\circ f^2|_\Delta\circ h:[0,1]\to[0,1]$ with $(h^{-1}\circ f^2|_\Delta\circ h)(0)=(h^{-1}\circ f^2|_\Delta)(\mathrm{P}_1)=h^{-1}(\mathrm{P}_1)=0$ and $(h^{-1}\circ f^2|_\Delta\circ h)(1)=(h^{-1}\circ f^2|_\Delta)(\mathrm{P}_3)=h^{-1}(\mathrm{P}_3)=1$, so $h^{-1}\circ f^2|_\Delta\circ h:[0,1]\to[0,1]$ is an order preserving homeomorphism. Hence $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)\in\{0,+\infty\}$, by Lemma 2.1. We have the following cases:

- Case 1: $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)=+\infty$. By [1] we have $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)=\mathrm{ent}_{\mathrm{set}}(f^2|_\Delta)\leq\mathrm{ent}_{\mathrm{set}}(f^2)$, so in this case $\mathrm{ent}_{\mathrm{set}}(f^2)=+\infty$ which leads to $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$ by Remark 1.1.
- Case 2: $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)=0$. By Lemma 2.1, $h^{-1}\circ f^2|_\Delta\circ h=\mathrm{id}_{[0,1]}$ thus $f^2|_\Delta=\mathrm{id}_\Delta$. For all $t\in[0,1]$, by $f^2<t,t>=<t,t>$ and Remark 2.2 $g_t:[0,1]\to[0,1]$ with $f^2<t,x>=<t,g_t(x)>$ is a homeomorphism, hence $g_t^2:[0,1]\to[0,1]$ is an order preserving homeomorphism and $\mathrm{ent}_{\mathrm{set}}(g_t^2)\in\{0,+\infty\}$, using Lemma 2.1, we have the following sub-cases:
  - Sub-case 2-1: $\mathrm{ent}_{\mathrm{set}}(g_t^2)=0$ for all $t\in[0,1]$. By Lemma 2.1 for all $t\in[0,1]$ in this sub-case we have $g_t^2=\mathrm{id}_{[0,1]}$, thus for all $x\in[0,1]$ we have $f^4<t,x>=f^2<t,g_t(x)>=<t,g_t^2(x)>=<t,x>$, so in this sub-case $f^4=\mathrm{id}_{\mathbf{A}}$.
  - Sub-case 2-2: $\mathrm{ent}_{\mathrm{set}}(g_t^2)=+\infty$ for some $t\in[0,1]$. By Remark 1.1 for all $n\geq1$ there exist $x_1,\ldots,x_n\in[0,1]$ such that $\{g_t^{2k}(x_1)\}_{k\geq1},\ldots,\{g_t^{2k}(x_n)\}_{k\geq1}$ are n pairwise disjoint one-to-one sequences, however for all $k\geq1$ and $i\in\{1,\ldots,n\}$ we have $f^{2k}<t,x_i>=<t,g_t^{2k}(x_i)>$, thus
  $$\{f^{2k}<t,x_1>\}_{k\geq1},\ldots,\{f^{2k}<t,x_n>\}_{k\geq1}$$
  are n pairwise disjoint one-to-one sequences, so $\mathrm{ent}_{\mathrm{set}}(f^2)\geq n$ which leads to $\mathrm{ent}_{\mathrm{set}}(f^2)=+\infty$ and $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$ by Remark 1.1.

Using the above cases (and sub-cases) the proof is completed.

3. Set-theoretical entropy of homeomorphisms of lexicographic ordered unit square

Consider lexicographic order $\preceq$ on $[0,1]\times[0,1]$, such that for $<x,y>,<z,w>\in[0,1]\times[0,1]$, let $<x,y>\preceq<z,w>$ "$x<z$" or "$x=z$ and $y\leq w$". Suppose $\mathbf{O}:=[0,1]\times[0,1]$ equipped with lexicographic order topology. In this section we compute set-theoretical entropies of homeomorphisms on $\mathbf{O}$.

Remark 3.1. In homeomorphism $f:\mathbf{O}\to\mathbf{O}$ for all $t\in[0,1]$ there exists $s\in[0,1]$ such that $f(\{t\}\times[0,1])=\{s\}\times[0,1]$ in addition $g:[0,1]\to[0,1]$ with $f<t,x>=<s,g(x)>$ is a homeomorphism. Moreover exactly one of the following conditions occurs [2]:

- $f(\mathrm{P}_i)=\mathrm{P}_i$, $f(\mathrm{L}_i)=\mathrm{L}_i$ $(i=1,2,3,4)$, and $f:\mathbf{O}\to\mathbf{O}$ is order-preserving,
- $f(\mathrm{P}_1)=\mathrm{P}_3$, $f(\mathrm{P}_2)=\mathrm{P}_4$, $f(\mathrm{P}_3)=\mathrm{P}_1$, $f(\mathrm{P}_4)=\mathrm{P}_2$, $f(\mathrm{L}_1)=\mathrm{L}_3$, $f(\mathrm{L}_2)=\mathrm{L}_4$, $f(\mathrm{L}_3)=\mathrm{L}_1$, $f(\mathrm{L}_4)=\mathrm{L}_2$, and $f:\mathbf{O}\to\mathbf{O}$ is anti-order-preserving.

Theorem 3.2. For homeomorphism $f:\mathbf{O}\to\mathbf{O}$ the following statements are equivalent:

- $\mathrm{ent}_{\mathrm{set}}(f)>0$,
- $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$,
- $f^2\neq\mathrm{id}_{\mathbf{O}}$,

i.e., $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if $f^2=\mathrm{id}_{\mathbf{O}}$.

Proof. Suppose $\mathrm{ent}_{\mathrm{set}}(f)>0$. By Remark 1.1, we have $\mathrm{ent}_{\mathrm{set}}(f^2)>0$. By Remark 3.1 for order-preserving homeomorphism $f^2:\mathbf{O}\to\mathbf{O}$ we have $f^2(\mathrm{P}_i)=\mathrm{P}_i$, $f^2(\mathrm{L}_i)=\mathrm{L}_i$ $(i=1,2,3,4)$. Using similar method described in the proof of Lemma 2.1 we have: $\mathrm{ent}_{\mathrm{set}}(f^2)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f^2)=0$ if and only if $f^2=\mathrm{id}_{\mathbf{O}}$. Use Remark 1.1 to complete the proof.

Example 3.3. Define $\varphi,\mu:[0,1]\to[0,1]$ with $\varphi(t):=1-t\,(t\in[0,1])$ and

$$\mu(t) := \begin{cases} 1-2t^2 & t \in [0, \frac{1}{2}], \\ \sqrt{\frac{1-t}{2}} & t \in [\frac{1}{2}, 1], \end{cases}$$

also consider $f, g : [0,1] \times [0,1] \to [0,1] \times [0,1]$ with $f < s,t >=< \varphi(s), \varphi(t) >$, $g < s,t >=< \mu(s), \mu(t) >$ (for $< s,t > \in [0,1] \times [0,1]$). Then:

- $f, g : \mathbf{A} \to \mathbf{A}$ and $f, g : \mathbf{O} \to \mathbf{O}$ are homeomorphisms,
- $f^2 = g^2 = \mathrm{id}_{[0,1]\times[0,1]}$ thus $\mathrm{ent}_{\mathrm{set}}(f) = \mathrm{ent}_{\mathrm{set}}(g) = 0$,
- $(g \circ f)^2(\frac{3}{4}) = \frac{31}{32}$ and $\mathrm{ent}_{\mathrm{set}}(g \circ f) = +\infty$ by Theorem 3.2.

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