Bifurcation in discrete prey-predator model

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Abstract: The dynamics of discrete-time prey-predator model are studied and investigated. The model has four fixed points. The origin fixed point is always exists while the others are exist under some conditions. The conditions that required achieving local stability of all fixed points are also set. The results indicate that the model has a flib bifurcation which found by varying the prey intrinsic growth parameter via pray and predator populations, respectively. Finally, numerical simulations not only illustrate our results, but also exhibit the complex dynamic behavior and chaotic.

Keywords: Discrete model, bifurcation theory, Competition.

1-Introduction:

Competition is an interaction between organisms or species in which both species are harmed. Competition may be for territory which is directly related to food resources. Some interesting phenomena have been found from the study of practical competition models. Hsu et al. [1] concerned with the growth of two predator species competing exploitatively for the same prey population. The predators feed on the prey with a saturating functional response to their prey density. The existence of species in the real world is not a lone so that the interaction, mutualism and competitive mechanisms are taken place. For that researchers have been investigated extensively in the recent years. They formed their models by using a set of differential equations [3,4,5]. Many authors have been carried out studying the chaotic dynamics that occur in multispecies continues time as well as discrete time prey-predator models [6,7,8]. In [9,10,11,12,13] authors have been given a modification of the system using nonlinear difference equations or partial differential equations .

Another example of competition is in Holt et al. [2]. They focused on the competition between two or more victim species that share a natural enemy. They also reviewed empirical examples of apparent competition in phytophagous insect hosts attacked by polyphagous parasitoids and they developed models of apparent competition in host-parasitoid systems. They found that the apparent competition is particularly likely in insect assemblages because parasitoids can limit their hosts to levels at which resource competition is unimportant.

This paper is organized as follows: in Section 2, the discrete prey–predator model is formulated and investigated, and then the conditions of existence and local stability of its fixed points are derived. In Section 3, we discussed that the model undergoes flip bifurcation in the interior ${R^2}_+$, by varying some values of parameters. Also, the numerical simulations are done to confirm the analytic results, such as the local stability as well as the bifurcation diagrams, phase portraits. Finally, in section 4 the conclusions are drawn.

2-The model and the analysis of its fixed points:

Consider the following discrete prey-predator model

$$\begin{cases} x_{t+1} = ax_t\left(1 - \frac{x_t}{1+y_t}\right) \\ y_{t+1} = cy_t\left(1 - \frac{y_t}{1+x_t}\right) \end{cases} \qquad (1)$$

This model describes the interactions between two populations with the initial conditions x(0)>0 , y(0)>0 , where the x(t) and y(t) denote the number of prey and the number pf predator at time t, respectively. The parameters a and c are the growth rate of the two species, respectively. The possible fixed points are obtained by solving the following algebraic equations:

$$\begin{cases} x = ax\left(1 - \frac{x}{1+y}\right) \\ y = cy\left(1 - \frac{y}{1+x}\right) \end{cases}$$

With simple computation we get the following fixed points:

1) $e_1 = (0,0)$ is the origin fixed point which is always exists.

2) $e_2 = (r_1, 0)$, where $r_1 = \frac{a-1}{a}$, is the first axial fixed point which means the prey population exist with absence of predator one.

3) $e_3 = (0, r_2)$, where $r_2 = \frac{c-1}{c}$, is the second axial fixed point which means the predator population exist with absence of prey one.

4) $e_4 = (x^*, y^*) = (\frac{(1-a)(2c-1)}{1-(a+c)}, \frac{(1-c)(2a-1)}{1-(a+c)})$ is the unique positive fixed point which exist if and only if $a, c > 1$.

For studying the stability of each fixed point we shall obtain the variation matrix and its characteristic equation. In general with (x, y) is a fixed point of model (1), the Jacobian matrix at (x, y) can be written as;

$$J(x,y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$$

Where

$$j_{11} = a - \frac{2ax}{1+y}$$

$$j_{12} = \frac{ax^2}{(1+y)^2}$$

$$j_{21} = \frac{cy^2}{(1+x)^2}$$

$$j_{22} = c - \frac{2cy}{1+x}$$

and characteristic equation of $J((x, y))$ is:

$$F(\lambda) = \lambda^2 + P\lambda + Q \qquad (2)$$

Where $P = c + a - \left(\frac{2cy}{1+x} + \frac{2ax}{1+y}\right)$ and

$$Q = \left(-\frac{2ax}{y+1}+a\right)\left(-\frac{2cy}{x+1}+c\right) - \frac{acx^2y^2}{(x+1)^2(y+1)^2}$$

Hence the system (1) is a dissipative system if $\left|\left(-\frac{2ax}{y+1}+a\right)\left(-\frac{2cy}{x+1}+c\right) - \frac{acx^2y^2}{(x+1)^2(y+1)^2}\right| < 1$ [12].

Let λ_1 and λ_2 be the two roots of equation(2), which are called the eigenvalues of the Jacobian matrix at any point. We recall some definitions of topological types for a fixed point. A fixed point is called a sink point if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink point is locally asymptotically stable. A fixed point is called a source point if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source point is locally unstable. A fixed point is called a saddle point if $|\lambda_1| > 1$ and $|\lambda_2| < 1$(or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). And a fixed point is called non-hyperbolic point if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$ [12]. The next propositions give the behavior dynamics of the fixed point e_1 as well as e_2 and e_3.

Proposition 2.1: The origin fixed point e_1 is:

a) Sink point if $a < 1$ and $c < 1$;
b) Source point if $a > 1$ and $c > 1$;
c) Non-hyperbolic point if $a = 1$ or $c = 1$;
d) Saddle point otherwise.

Proof: It is clear that the Jacobian matrix at e_1 is given as follows:

$$J_{e_1} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

Obviously, the eigenvalues of the J_{e_1} are $\lambda_1 = a$ and $\lambda_2 = c$, therefore all results can be obtained.

Proposition 2.2: For the fixed points e_2 and e_3 we have:

1- For the prey axial fixed point e_2 is:
a) Sink point if $1 < a < 3$ and $c < 1$;
b) Source point if $a > 3$ and $c > 1$;
c) Non-hyperbolic point if either $a = 1$or 3 or $c = 1$;
d) Saddle point otherwise.

2- For the predator fixed points there exist at least four different topological types these are:
a) Sink point if $a < 1$ and $1 < c < 3$;
b) Source point if $a > 1$ and $c > 3$;
c) Non-hyperbolic point if $a = 1$ either $= 1 or 3$;
d) Saddle point otherwise.

Proof: It is clear that the Jacobian matrices at $e_2$and e_3 are given by:

$$J_{e_2} = \begin{pmatrix} a-2ar_1 & a{r_1}^2 \\ 0 & c \end{pmatrix}$$

$$J_{e_3} = \begin{pmatrix} a & 0 \\ c{r_1}^2 & c-2cr_2 \end{pmatrix}$$

Hence, the eigenvalues of the J_{e_2} are $\lambda_1 = 2 - a$ and $\lambda_2 = c$ while the eigenvalues of the J_{e_3} are $\lambda_1 = a$ and $\lambda_2 = 2 - c$ therefore all results can be obtained, respectivaly.

Before studying the behavior of the unique positive fixed point e_4 , we need the following Lemma which appeared in[13]

Lemma 2.3 : Let $F(\lambda) = \lambda^2 + P\,\lambda + Q$. Suppose that $F(1) > 0, \lambda_1$ and λ_2 are the two roots of $F(\lambda) = 0$. Then
(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;
(ii)$|\lambda_1| < 1$ and$|\lambda_2| > 1$ (or$|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;
(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$and $Q > 1$;
(iv) $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$.
Proof: see [13].

In order to discuss the dynamics behavior of the positive fixed point e_4, we need the Jacobian matrix at e_4 which is given by

$$J(x,y) = \begin{pmatrix} 2-a-\lambda & \frac{(a-1)^2}{a} \\ \frac{(c-1)^2}{c} & 2-c-\lambda \end{pmatrix}$$

Where P and Q in equation (2) are

$$P = a - 4 + c \text{ and}$$

$$Q = (a-2)(c-2) - \frac{(a-1)^2(c-1)^2}{ac}$$

Now, the next proposition gives the dynamics of the positive fixed point.

Proposition 2.4: The unique positive fixed pointe_4 is:

1- Sink point if and only if the $a \in (A, \infty) \cap I \cap [(0, B_2) \cup (B_1, \infty)]$

2- Source point if and only if the $a \in (A, \infty) \cap I \cap (B_2, B_1)$

3- Saddle point if $a \in (A, \infty) \cap [(0, \min\{b_1, b_2\}) \cup (\max\{b_1, b_2\}, \infty)]$.

4- Non-hyperbolic point if $a \in (A_1, \infty)$ and either $a \neq 4 - c \; or \; a \neq 2 - c$;

Where

$A = \frac{2-c+\sqrt{(c-2)^2+4(c-1)}}{2}$

$,B_1 = \frac{2-c+\sqrt{(c-2)^2-4(c-1)^2}}{2}$,

$$B_2 = \frac{2 - c - \sqrt{(c-2)^2 - 4(c-1)^2}}{2}$$

$I = (\min\{b_1, b_2\}, \max\{b_1, b_2\})$,

$b_1 = \frac{-k_1 + \sqrt{{k_1}^2 - 4k_2}}{2}$ and $b_2 = \frac{-k_1 - \sqrt{{k_1}^2 - 4k_2}}{2}$ while $k_1 = \frac{c^2-5c-2}{c+1}$ and $k_2 = \frac{(c-1)^2}{c+1}$.

Proof: We will apply Lemma 2.3. Therefore:

$F(1) = 1 + P + Q = 1 + a - 4 + c + (a-2)(c-2) - \frac{(a-1)^2(c-1)^2}{ac} > 0$

That implies $a^2 + (c-2)a - (c-1) > 0$. Thus $F(1) > 0$ if and only if $a \in (A_1, \infty)$.

Now, we have to show that $F(-1) > 0$ and $Q < 1$. So that, we have the following steps:

$F(-1) = 1 - a + 4 - c + (a-2)(c-2) - \frac{(a-1)^2(c-1)^2}{ac} > 0$

That implies $a^2 + \frac{(c^2-5c-2)}{c+1}a + \frac{(c-1)^2}{c+1} < 0$. Therefore $F(-1) > 0$ if and only if when

$a \in I$

It is clear that $Q = (a-2)(c-2) - \frac{(a-1)^2(c-1)^2}{ac} < 1$ if and only if $a^2 - (2-c)a + (c-1)^2 > 0$ therefore $Q < 1$ if $a \in (\infty, B_2) \cup (B_1, \infty)$

According to the Lemma 2.3(1), $e_4$is sink when

$$a \in (A_1, \infty) \cap I \cap [(0, B_2) \cup (B_1, \infty)]$$

The proof of the other cases can be easily obtained.

3-Numerical simulation:

To provide some numerical evidence for the qualitative dynamic behavior of the model (1),so that at different set of values the local behavior of the all fixed points are investigated numerically. For the fixed point e_1 we choose the value of $a = 0.7$ and $c = 0.8$ as well as we choose the values $a = 1.7$ and $c = 0.8$ and $a = 0.7$ and $c = 1.8$ for the fixed points e_2 and e_3, respectively . Figures 1, 2, and 3 indicate the stability of e_1, e_2 and e_3 with the initail value (0.6,0.5). For the positive fixed point the values of $a = 1.8$ and $c = 1.2$ are chosen that satisfy the condition 1 in proposition 2.4. Figure 4 shows the local stability of the e_4=(0.55,0.26) with initial value (0.6,0.5).

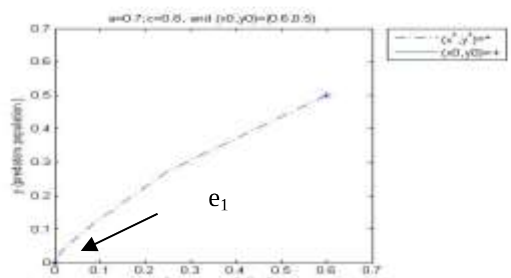


Figure 1: This figure shows the stability of e_1 according to the proposition 2.1.

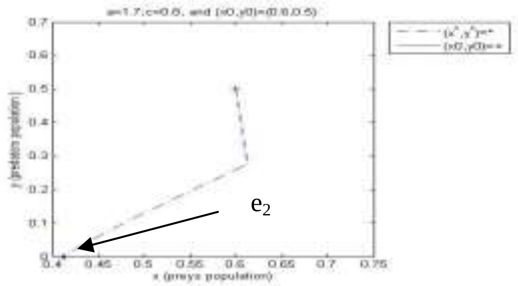


Figure 2:The stability of e_2 under the conditions of the proposition 2.2

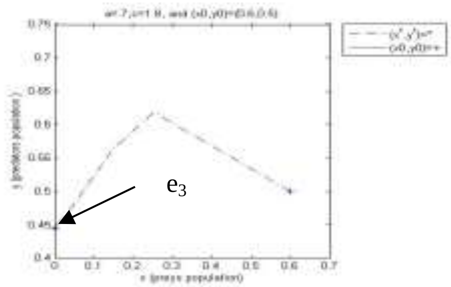


Figure 3: This figure shows the stability of e_3 according to the proposition 2.2

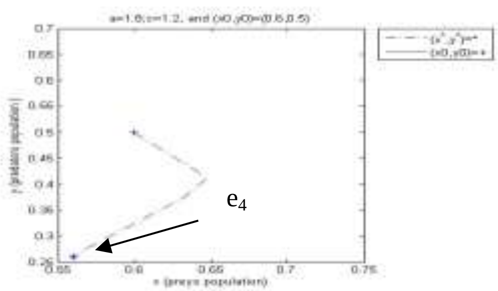
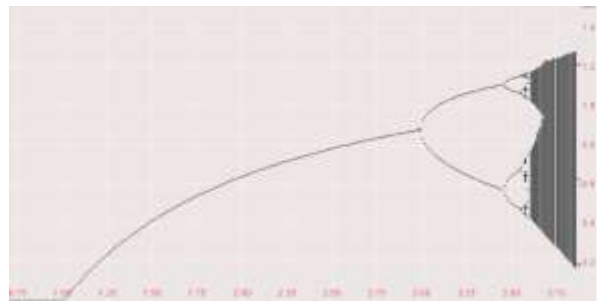


Figure 4: The stability of the positive fixed point e_4 according to the proposition 2.4

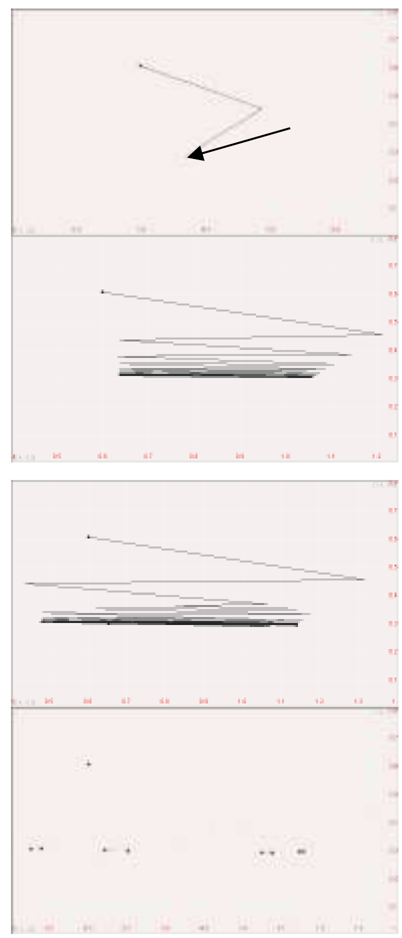
In different point of view, we study the phase portrait of the model (1) when we change only the parameter a via prey population and fix the others. To study the behavior of the model (1) when the parameter varied in the interval [0.75,3.95] one can consider the initial condition (0.6, 0.6) which is varied in the basin of attraction of positive fixed point e_4. When the control parameter varies, the stability of a periodic solution may be lost through various types of bifurcations and it gives the stable, period-2, period-4,perod-8,period-16,period-32 then chaotic

Now, without loss of generality we fix the parameters$c = 1.2$, and we assume that a is varied inside the interval$[0.75,3.95]$. The phase portraits are considered in the Figures 5,6,7,8,and 9:



a

Figure5: Bifurcation diagram for system (1) versus a via prey population.



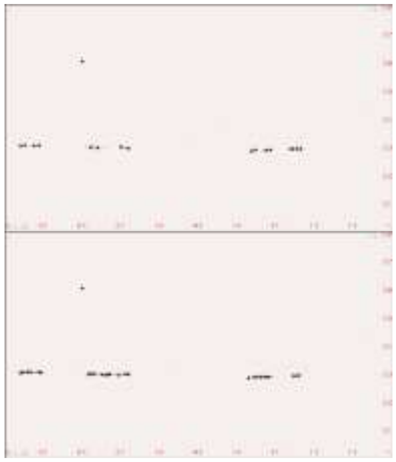


Figure6: These phase diagrams when $a = 2, 3.24, 3.5001, 3.544, 3.556, 3.5587$, respectively.

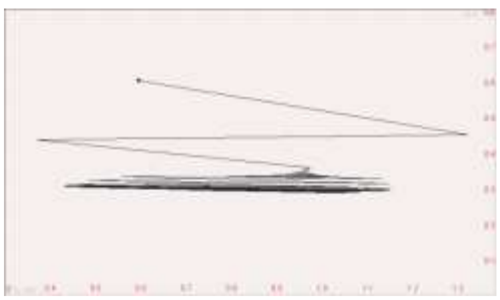


Figure7 : These phase diagrams gives the chaotic when $a = 3.564$.

The second numerical case starts when we will study the phase portrait of the model (1)as only the parameter a via predator population and fix the others. To study the behavior of the model (1) when the parameter varied in the interval [0.9,3.95] one can consider the initial condition (0.6, 0.6) situated in the basin of attraction of fixed point e_4. When the control parameter varies, the stability of a periodic solution may be lost through various types of bifurcations and it gives the stable, period-2, period-4 then chaotic.

Now, without loss of generality we fix the parameters $c = 1.2$, and we assume that $a \in [0.9,3.95]$. The phase portraits are considered in the following Figures:

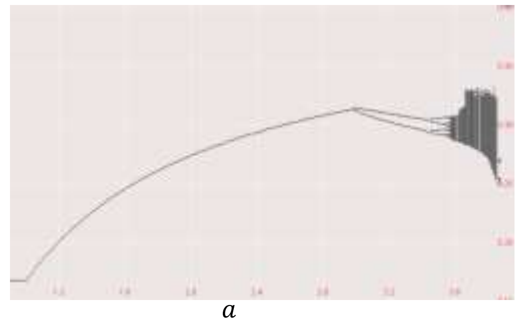


Figure 8: Bifurcation diagram for system (1) versus a via predator population.

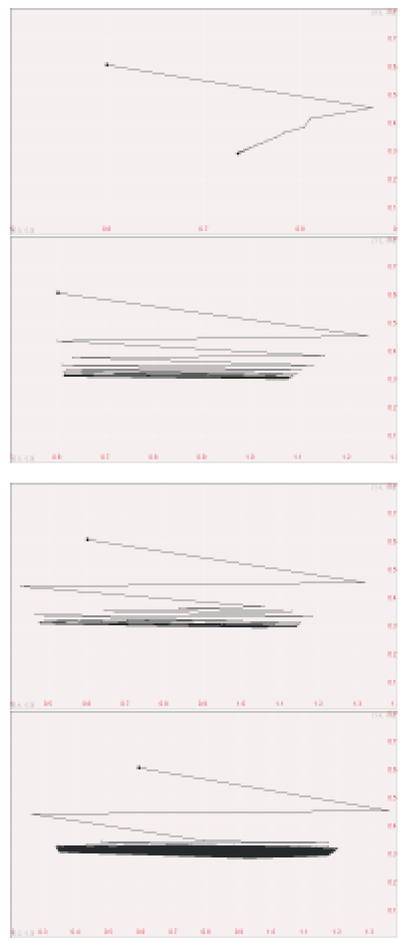


Figure 9: These phase diagrams give when $a = 2.33, 3.3, 3.507, 3.66$, respectively.

4-Conclusion:

In this paper, the local stability of all possible fixed points of a two dimensional discrete time prey-predator model has been studied and discussed. The chaotic dynamics and bifurcation of the model have been investigated. Basic properties of the model have been analyzed by means of phase portrait, and bifurcation diagrams. Under certain parametric conditions, the interior fixed point enters a flib bifurcation phenomenon. This could be very useful for the biologists as well as mathematicians who work with discrete-time prey–predator models.

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