On T-extending modules

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Abstract

In this paper we introduce the concepts of the T-direct sum and T-extending modules and we give some basic properties of these types of modules. Also we define the relations α_T and β_T on the set of submodules containing T of a module M and we give some basic properties.
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1- Introduction

In this paper, all rings are associative with identity and all modules are unitary left R-modules. Recall that a submodule A of an R-module M is essential submodule of M{denoted by $A \leq_e M$}, if for every $B \leq M$, $A \cap B = 0$ implies that $B = 0$.
A submodule B of a module M is called complement for a submodule A of M if it is maximal with respect to the property that $A \cap B = 0$. More details about essential submodules and complement can be found in [1] .
A module M is an **extending module** (denoted by CS- module), if every submodule of M is essential in a direct summand of M , see [2, 3].
Let M be a module. Recall the following relation on the set of submodules of M : **A α B** if there exists a submodule C of M such that $A \leq_e C$ and $B \leq_e C$, see [4]. Let M be a module. Recall the following relation on the set of submodules of M: **A β B** if $A \cap B \leq_e A$ and $A \cap B \leq_e B$, see [4]. In [5], the authors introduced the definition of T–essential (complement) submodules as follows: Let $T \lneq M$, a submodule A of M is called T–essential submodule of M {denoted by $A \leq_{Tes} M$}, provided that $A \nleq T$ and for each submodule B of M , $A \cap B \leq T$ implies that $B \leq T$. A submodule B of M is called a T –complement for a submodule A in M if B is maximal with respect to the property that $A \cap B \leq T$. In [6], we introduce the definition of T-closed submodules as follows: Let T , A and B be submodules of a module M . A is called a T–closed submodule of M (denoted by $A \leq_{Tc} M$), if $A \leq_{Tes} B$ implies that $A + T = B$, for every submodule B of M.
In section 2 , we will introduce the definition of T-direct sum modules as follows : Let T, A and B be submodules of a module M. M is called T-direct sum of A and B (denoted by $M = A \oplus_T B$). If $M = A + B$ and $A \cap B \leq T$.
In this case, each of A and B is called a T-direct summand of M .We prove that Let T, A and B be submodules of a distributive module M. If B is a T-complement for A in M , then $A \oplus_T B \leq_{Tes} M$, see proposition (2.11). Also we introduce the definition of T-extending modules as follows:
Let T be a submodule of a module M. We say that M is **T-extending module** (denoted by T-CS modules) if every submodule of M which contains T is T-essential in every T-closed submodule of M which contains T is a T-direct summand of M, see proposition (2.15).
In section three , we will define the following relation : Let A and B be submodules of a module M with $T \leq A$ and $T \leq B$. We say that **A α_T B** if there exists a submodule C such that $A \leq_{Tes} C$ and $B \leq_{Tes} C$.
Also we define the following relation : Let A and B be submodules of a module M with $T \leq A$ and $T \leq B$. We say that **A β_T B** if $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. We prove that : The β_T is an equivalence relation , see proposition (3.10).

2. The T-extending modules

In this section , we will introduce the concepts of the **T-direct sum** and **T-extending modules** and we illustrate it by some examples. We also give some basic properties of these type of modules.

Definition (2.1): Let T, A and B be submodules of a module M. M is called **T-direct sum** of A and B (denoted by $M = A \oplus_T B$). If $M = A + B$ and $A \cap B \leq T$. In this case , each of A and B is called a T-direct summand of M .

Let M be a module . Clearly that every direct summand of M is a T-direct summand. And when T = 0, a submodule A of M is a T-direct summand of M if and only if A is a direct summand of M.

Examples (2.2):

(1) Consider the module Z as Z-module and let T = 6Z . Clearly that $Z = 2Z \oplus_T 3Z$. But 2Z is not a direct summand of Z . Now let T = 4Z . $2Z \cap 3Z = 6Z \nleq 4Z$, then Z is not 4Z-direct sum of 2Z and 3Z.

(2) The Z_{12} as Z-module. Let $T = \{\bar{0},\bar{6}\}$, $A = \{\bar{0},\bar{2},\bar{4}, \bar{6},\bar{8},\overline{10}\}$ and $B = \{\bar{0},\bar{3},\bar{6},\bar{9}\}$. One can easily show that A is $\{\bar{0},\bar{6}\}$-direct summand of Z_{12} , and A is not direct summand of Z_{12} .

Proposition (2.3): Let T, A and B be submodules of a module M such that $\frac{M}{T} = \frac{A}{T} \oplus \frac{B}{T}$. Then $M = A \oplus_T B$.

Proof: suppose that $\frac{M}{T} = \frac{A}{T} \oplus \frac{B}{T}$. Then $M = A + B$ and $\frac{A}{T} \cap \frac{B}{T} = \frac{A \cap B}{T} = 0$ and hence $A \cap B = T$. Thus $M = A \oplus_T B$

Note: The converse of proposition is not true in general , for example . Consider the module Z as Z-module and

D is a T-direct summand of M. Thus M is a T-extendin .

Remark (2.16): Let T be a submodule of M. If T = 0 then M is T-extending if and only if M is extending.

Proof: Clear.

let T = A = 4Z , B = 3Z. Cleary that M =A $\oplus_T$ B . But A ∩ B = 12Z ≠ T. Thus $\frac{M}{T}$ is not the direct sum of $\frac{A}{T}$ and $\frac{B}{T}$.

Remark (2.4): Let T , A and B be submodules of a module M such that A ≤ B ≤ M and T ≤ B . If A is a T-direct summand of M , then A is a T-direct summand of B.

Proof: Let A be a T-direct summand of M, then M = A $\oplus_T$ C, for some submodule C of M. Since A ≤ B , then by modular law, B = M ∩ B = (A $\oplus_T$ C) ∩ B = A $\oplus_T$ (C ∩ B). Thus A is a T-direct summand of B.

A module M is called a **distributive module** if A ∩ (B + C) = (A ∩ B) + (A ∩ C) , for all submodules A , B and C of M. See [7].

Lemma (2.5): [8] Let A , B and C be are submodules of a module M . Then the following statement are equivalent :

(1) A ∩ (B + C) = (A ∩ B) + (A ∩ C).

(2) A + (B ∩ C) = (A + B) ∩ (A + C).

Proposition (2.6): Let T, A and B be submodules of a distributive module M such that M = A $\oplus_T$ B , then $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$.

Proof: Assume that M = A $\oplus_T$ B .Then $\frac{M}{T} = \frac{A+B+T}{T} = \frac{A+T}{T} + \frac{B+T}{T}$. Since A ∩ B ≤ T , then (A ∩ B) + T ≤ T . Since M is a distributive module, the (A + T) ∩ (B + T) = (A ∩ B) + T ≤ T, by lemma (2.5). But T ≤ (A + T) ∩ (B + T) , therefore (A + T) ∩ (B + T) = T . Hence $\frac{A+T}{T} \cap \frac{B+T}{T} = 0$. Thus $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$.

Proposition (2.7): Let T , A and B be submodules of a module M such that A ≤ B . If A is T-direct summand of B and B is T-direct summand of M , then A is T-direct summand of M.

Proof: Suppose that A is T-direct summand of B , then B = A $\oplus_T$ C , where C be a submodule of B . Since B is T-direct summand of M, then M = B $\oplus_T$ D , where D be a submodule of M. Implies that M = (A $\oplus_T$ C) $\oplus_T$ D . Hence M = (A + C) + D = A + (C + D) and A ∩ (C ∩ D) = (A ∩ C) ∩ D ≤ T. Then M =A $\oplus_T$ (C $\oplus_T$ D). Thus A is T-direct summand of M.

Proposition (2.8): Let T, A and B be submodules of a distributive module M such that M =A $\oplus_T$ B .Then B + T is a T-complement for A + T in M.

Proof: Suppose that M is a distributive module and M = A $\oplus_T$ B . Then by (2.6) , $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$. Thus B + T is a T-complement for A+T in M, by [9,Coro.3.5,p. 907]

Corollary (2.9): Let T, A and B be submodules of a distributive module M such that M = A $\oplus_T$ B . Then A + T is T-closed submodule of M.

Proof: Assume that M = A$\oplus_T$ B, then $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$, by (2.6). Then $\frac{A+T}{T}$ is closed in $\frac{M}{T}$ by [1] . Thus A + T is T-closed submodule of M, by [6 , Prop. 2.9 , p. 1684].

Corollary (2.10): Let T , A and B be submodules of adistributive module M such that T ≤ A and M =A$\oplus_T$ B . a T-direct summand of M. we prove that : Let M be a module. Then M is T-extending module if and only if.

Proof: Let M = A $\oplus_T$ B, then A is a T-closed in M , by (2.9). Since T ≤ A , then $\frac{A}{T}$ is closed submodule of $\frac{M}{T}$, by [6, Coro. 2.10 , p. 1684].

Proposition (2.11): Let T , A and B be submodules of a distributive module M . If B is a T-complement for A in M , then A $\oplus_T$ B $\leq_{Tes}$ M.

Proof: Let B be a T-complement for A in M, then A ∩ B ≤ T. Let C be a submodule of M such that (A $\oplus_T$ B) ∩ C ≤ T. Since M is a distributive module, then (A∩C) $\oplus_T$ (B∩ C) ≤ T and A ∩ (B $\oplus_T$ C) = (A ∩ B) $\oplus_T$ (A ∩ C) ≤ T. But B is maximal with respect to property that A ∩ B ≤ T, therefore B + C = B. Implies that C ≤ B. Hence C = C ∩ B ≤ T .Thus A $\oplus_T$ B $\leq_{Tes}$ M.

Corollary (2.12): Let T, A and B be submodules of a distributive module M. If $\frac{B}{T}$ is a relative complement for $\frac{A}{T}$ in $\frac{M}{T}$ then A $\oplus_T$ B $\leq_{Tes}$ M.

Proof: Suppose that $\frac{B}{T}$ is a relative complement for $\frac{A}{T}$ in $\frac{M}{T}$, then B is a T-complement for A in M, by [9, Prop. 3.4, p. 907]. Hence A $\oplus_T$ B $\leq_{Tes}$ M, by (2.11).

Proposition (2.13): Let A, B, C and D be submodules of a distributive module M such that T, A, C ≤ B . If M = B $\oplus_T$ D and C is a T-complement of A in B , then C $\oplus_T$ D is a T-complement for A in M.

Proof: Let M = B $\oplus_T$ D and C be a T-complement for A in B. Then M = B + D, B ∩ D ≤ T and A ∩ C ≤ T. Since C ≤ B, then C ∩ D ≤ B ∩ D ≤ T. As A ≤ B, then A ∩ D ≤ B ∩ D ≤ T. But M is a distributive module, hence we obtain A∩ (C $\oplus_T$ D) = (A ∩ C) $\oplus_T$ (A ∩ D) ≤ T .Now let L be a submodule of M such that C$\oplus_T$ D ≤ L and A ∩ L≤ T. Then (L ∩ A) ∩ B = (A ∩ L) ∩ B ≤ T. But C is maximal with respect to the property that A ∩ C ≤ T, therefore, C = L ∩ B. Thus L = M ∩ L = (B $\oplus_T$ D) ∩ L = (B ∩ L) $\oplus_T$ (D ∩ L) = C $\oplus_T$ D. Which means C $\oplus_T$ D is a T-complement for A in M.

We introduce the following definition

Definition (2.14): Let T be a submodule of a module M. We say that M is **T-extending module** (denoted by T-CS modules) if every submodule of M which contains T is T-essential in a T-direct summand of M.

Proposition (2.15): Let M be a module. Then M is T-extending module if and only if every T-closed submodule of M which contains T is a T-direct summand of M.

Proof: Suppose that M is a T-extending module and let A be a T-closed submodule of M such that T ≤ A . Since M is a T-extending module , then there exist a T-direct summand D of M such that A $\leq_{Tes}$ D . But A is a T-closed submodule of M , therefore A + T = D .Thus A = D.

Conversely, let A be a submodule of M such that T ≤ A . So there exist a T-closed submodule D in M such that A $\leq_{Tes}$ D, by [6, Prop. 2.12, P.1684]. By our assumption way D is T-essential in Z_P^{∞} then A and D are T-essential in Z_P^{∞}. Thus $Z_p^{\ n}\ \alpha_T\ Z_p^{\ m}$.

Proposition (2.17): Every T-direct summand contain T of a distributive and T-extending module is T-extending module.

Proof: Let M be a distributive and T-extending module such that $M = A \oplus_T B$ and $T \leq A$, where T , A and B are submodules of M. Let C be a T-closed submodule in A such that $T \leq C$. Since A is a T-direct summand of M, then A is a T-closed submodule of M, by (2.9). Thus C is a T-closed in M , by [6, Th. 2.14 , p. 1684]. But M is a T-extending , therefore C is a T-direct summand of M by (2.15). Since $C \leq A$, then C is a T-direct summand of A , by (2.4).

Examples (2.18):

(1) Consider the module Z_6 as Z-module and let $T =\{\bar{0},\bar{2},\bar{4}\}$. Then Z_6 and $\{\bar{0},\bar{2},\bar{4}\}$ are they only submodules of Z_6 that containing T. Since $\{\bar{0},\bar{2},\bar{4}\}$ is a T-essential and a T-direct summand of Z_6 and Z_6 is a T-essential of Z_6 .Then Z_6 is T-extending module.

(2) Consider the module Z as Z-module . Let T = 2Z , then Z and 2Z are they only submodules of Z that containing T . Since 2Z is T-essential in 2Z and 2Z is T-direct summand of Z .Then Z is 2Z-extending module.

(3) The module $M = Z_8 \oplus Z_2$ as a Z-module . It's known that M is not extending module , by [10, ex. (2.4.18). Ch.2] . Hence M is not {0}-extending module. Now let $T = \{\bar{0},\bar{2},\bar{4},\bar{6}\} \oplus Z_2$. Since M and T are the only submodules that containing T , then one can easily check that M is a T-extending module.

Proposition (2.19): Let T be a submodule of a module M. If $\frac{M}{T}$ is extending module, then M is a T-extending module.The converse is true if M is a distributive module.

Proof: Let A is a submodule of M such that $T \leq A$. Since $\frac{M}{T}$ is an extending module , then there exist a direct summand $\frac{B}{T}$ of $\frac{M}{T}$ such that $\frac{A}{T} \leq_e \frac{B}{T}$ .Therefore $A \leq_{Tes} B$ by [5,Lem. 2.3, P. 17] and B is a T-direct summand of M, by (2.3). Thus M is a T-extending .

For the converse , Let M be a distributive module $\frac{A}{T}$ be a submodule of $\frac{M}{T}$. Since M is T-extending and A is a submodule of M , then there exist a T-direct summand B of M such that $A \leq_{Tes} B$. Thus $\frac{A}{T} \leq_e \frac{B}{T}$, by [5, Lem. 2.3, p. 17]. Hence $M = B \oplus_T B_1$, for some submodule B_1 of M . But M is a distributive module , therefore $\frac{M}{T} = \frac{B}{T} \oplus \frac{B1+T}{T}$, by proposition (2.6). So $\frac{B}{T}$ is a direct summand of $\frac{M}{T}$. Thus $\frac{M}{T}$ is extending.

Theorem (2.20): Let T and A be submodules of a T-extending module M such that $T \leq A$. If the intersection of A with any T-direct summand of M containing T is a T-direct summand of A then A is T-extending module.

Proof: Let M be a T-extending module and N be a submodule of A such that $T \leq N$, then there exist T-direct summand D of M such that $T \leq D$ and $N \leq_{Tes} D$. Since $N \leq A \cap D$, then $N \leq_{Tes} A \cap D$, by [5, Prop. 2.12 , P. 19]. By our $A \cap D$ is a T-direct summand of A . Thus A is T-extending module .

Then $\frac{A}{T}$ is a closed submodule of $\frac{M}{T}$. K of M which containing T and either $K \cap A \leq T$ or $K \cap B \leq T$ is a T-direct summand of M.

Proof: Assume K is T-closed of M such that $T \leq K$ and $T = K \cap A$. Since M is T-extending module , then K is a T-direct summand of M by , (2.15) .

Theorem (2.22): Let T , A and B be submodules of a module M such that $M = A \oplus_T B$ and $T \leq A \cap B$. If every T-closed submodule K of M which containing T and either $K \cap A \leq T$ or $K \cap B \leq T$ is a T-direct summand of M , then every T-complement containing T for A or B in M is T-direct summand of M and T-extending module .

Proof: Let K be a T-complement for A in M such that $T \leq K$. Then K is a T-closed submodule in M . by [6. Th. 2.18 , P. 1684]. But $K \cap A \leq T$, therefore by our a assumption K is a T-direct summand of M.

Let L be a T-closed submodule of K such that $T \leq L$. Then L is a T-closed in M , by [6. Th.2.14 , p. 1684]. Since $L \cap A \leq K \cap A \leq T$. Then by our assumption L is a T-direct summand of M and hence L is a T-direct summand of K , by (2.4) . Thus K is a T-extending of M.

Theorem (2.23): Let T , A and B be submodules of a module M such that $M = A \oplus_T B$ and $T \leq A \cap B$. If M is T-extending module, then every T-complement containing T for A or B in M is T-direct summand of M and T-extending module .

Proof: Suppose that M is T-extending module and let K is a T-complement for A in M contain T , then K is T-closed in M, by [6. Th. 2.18 , P. 1684]. Since $K \cap A \leq T$, then K is a T-direct summand of M , by (2.21). Thus K is T-extending module , by (2.22).

3- The relations α_T and β_T :

In this section we define the relations $\boldsymbol{\alpha_T}$ and β_T . Also we give some basic properties of these relations.

Definition (3.1): Let T be a submodule of a module M and let S_T be the set of submodules of M that containing T. Let A and B $\in S_T$. We say A $\boldsymbol{\alpha_T}$ B if there exists a submodule C such that $A \leq_{Tes} C$ and $B \leq_{Tes} C$.

Let M be a module and T = 0 . Then one can easily show that A α B if and only if A α_T B , for each submodules A and B of M.

Examples (3.2):

(1) The module Z_4 as Z-module. Let $T =\{\bar{0}, \bar{2}\}$, $A =\{\bar{0}, \bar{2}\}$ and $B = Z_4$. Since A and B are T-essential in Z_4, then A α_T B.

(2) The module Z_{12} as Z-module . Let $T = \{\bar{0},\bar{6}\}$, $A = \{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ and $B = Z_{12}$. Since B is T-essential in Z_{12} and clearly that $A \cap \{\bar{0},\bar{3},\bar{6},9\} = T$. But $\{\bar{0},\bar{3},\bar{6},9\} \nleq T$, therefore A is not T-essential in Z_{12} . Thus A is not relate to B by α_T

(3) The module Z_P^{∞} as Z-module. Let $T = (\frac{1}{pn} + Z)$, $A = (\frac{1}{pm} + Z)$ and $D = (\frac{1}{pr}+ Z)$, where n, m, r $\in$ Z and m, r > n .Let B be a submodule of Z_P^{∞} such that $A \cap B \leq T$.

Since Z_P^{∞} is a uniserial module , then either $A \leq B$ or $B \leq A$. If $A \leq B$, we get $A \cap B = A \leq T$ which is a contradiction . Thus $B \leq A$ and hence $A \cap B = B \leq T$. Thus A is T-essential submodule of Z_P^{∞}. By the same

$A \cap C \leq T$ implies $B \cap C \leq T$ and $B \cap D \leq T$ implies $A \cap D \leq T$, for each submodules C and D of M.

Proposition (3.3): Let T be a submodule of a module M. Then $A\ \alpha_T\ B$ if and only if $\frac{A}{T}\ \alpha\ \frac{B}{T}$, for each A and $B \in S_T$.

Proof: Let $A\ \alpha_T\ B$. Then there exists a submodule C of M such that $A \leq_{Tes} C$ and $B \leq_{Tes} C$. Then $\frac{A}{T} \leq_e \frac{C}{T}$ and $\frac{B}{T} \leq_e \frac{C}{T}$, by [5,Lem. 2.3, P. 17].. Thus $\frac{A}{T}\ \alpha\ \frac{B}{T}$.
Conversely, let $\frac{A}{T}\ \alpha\ \frac{B}{T}$, then there exists a submodule $\frac{C}{T}$ of $\frac{M}{T}$ such that $\frac{A}{T} \leq_e \frac{C}{T}$ and $\frac{B}{T} \leq_e \frac{C}{T}$. Then $A \leq_{Tes} C$ and $B \leq_{Tes} C$, by [5, Lem. 2.3, P. 17]. Thus $A\ \alpha_T\ B$.

Remark(3.4): The α_T is a reflexive and symmetric relation.

Proof: Clear.

Proposition (3.5): Let T be a submodule of a module M. Then M is T-extending if and only if for each submodule $A \in S_T$, there exists a T-direct summand $D \in S_T$ such that $A\ \alpha_T\ D$

Proof: ⇒) Suppose that M is T-extending, and let $A \in S_T$. Since M is T-extending, then there exists a T-direct summand $D \in S_T$ such that $A \leq_{Tes} D$, we want to show that there exists a submodule B of M such that $A \leq_{Tes} B$ and $D \leq_{Tes} B$. Let B = D, then $A \leq_{Tes} D$ and $D \leq_{Tes} D$. Thus $A\ \alpha_T\ D$.
⇐) Let $A \in S_T$, by our assumption, there exists a T-direct summand $D \in S_T$ such that $A\ \alpha_T\ D$. Thus there exists a submodule $B \in S_T$ such that $A \leq_{Tes} B$ and $D \leq_{Tes} B$. It is enough to show that B is a T-direct summand of M. Let $M = D \oplus_T D_1$, where D_1 is a submodule of M. Since $D \leq B$ then $M = B + D_1$. Since $D \cap D_1 \leq T$, then $(B \cap D) \cap D_1 \leq T$. But $D \leq_{Tes} B$, therefore $D_1 \cap B \leq T$. Hence $M = B \oplus_T D_1$. Claim that B = D. To show that, Let $b \in B$, then $b = d + d_1$, where $d \in D$ and $d_1 \in D_1$. So $b - d = d_1 \in (B \cap D_1) \leq T \leq D$. Hence $b = d + d_1 \in D$. This implies that B = D. Thus M is a T-extending module.

Proposition (3.6): Let T, A and B be submodules of a module M such that A and $B \in S_T$. If $A\ \alpha_T\ B$, then there exists a submodule C of M such that $\frac{C}{A}$ and $\frac{C}{B}$ are singular.

Proof: Assume that $A\ \alpha_T\ B$, then there exists a submodule $C \in S_T$ such that $A \leq_{Tes} C$ and $B \leq_{Tes} C$. Hence $\frac{A}{T} \leq_e \frac{C}{T}$ and $\frac{B}{T} \leq_e \frac{C}{T}$, by [5, Lem. 2.3, P. 17]. Now consider the following two short exact sequences:

$$0 \to \frac{A}{T} \xrightarrow{i} \frac{C}{T} \xrightarrow{\pi 1} \frac{C/T}{A/T} \to 0$$

$$0 \to \frac{B}{T} \xrightarrow{j} \frac{C}{T} \xrightarrow{\pi 2} \frac{C/T}{B/T} \to 0$$

Where i, j are inclusion map and π_1, π_2 are the natural epimorphisms. Since $\frac{A}{T} \leq_e \frac{C}{T}$ and $\frac{B}{T} \leq_e \frac{C}{T}$, then $\frac{C/T}{A/T}$ and $\frac{C/T}{B/T}$ are singular, by [2, Prop.1.20, P.31]. By the third isomorphis theorem, $\frac{C/T}{A/T} \cong \frac{C}{A}$ and $\frac{C/T}{B/T} \cong \frac{C}{B}$. Thus $\frac{C}{A}$ and $\frac{C}{B}$ are singular.

Definition (3.7): Let T be a submodule of a module M and let A and $B \in S_T$, then we say that $A\ \boldsymbol{\beta_T}\ B$ if $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$.

Theorem (2.21): Let T, A and B be submodules of a module M such that $M = A \oplus_T B$ and $T \leq A \cap B$. If M is T-extending module, if every T-closed submodule

Examples (3.8):
(1) The module Z_4 as Z-module. Let $T = \{\bar{0}\}$, $A = \{\bar{0},\bar{2}\}$ and $B = Z_4$, then $A \cap B = A \leq_{Tes} A$ and $A \cap B = A \leq_{Tes} B$. Thus $\{\bar{0},\bar{2}\}\ \beta_T\ Z_4$.
(2) Consider the module $Z_{P\infty}$ as Z-module. Let $T = (\frac{1}{p^n} + Z)$, $A = (\frac{1}{p^m} + Z)$ and where n, m ∈ Z and m > n and let $B = Z_{P^\infty}$. Since by (3.2-3), $A \leq_{Tes} Z_{P^\infty}$. and $A \leq_{Tes} A$. Then $Z_p^{\ m}\ \beta_T\ Z_{P^\infty}$.
(3) The module Z_{12} as Z-module. Let $T = \{\bar{0},\bar{6}\}$, $A = \{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ and $B = Z_{12}$, then $A \cap B = A \leq_{Tes} A$. But A is not T-essential in Z_{12}, by (3.2-2). Therefore $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ is not related to Z_{12} by β_T.

Properties (3.9): Let T, A and B be a submodules of a module M such that A, $B \in S_T$. Then $A\ \beta_T\ B$ if and only if $\frac{A}{T}\ \beta\ \frac{B}{T}$

Proof: ⇒) Suppose that $A\ \beta_T\ B$, then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Then $\frac{A \cap B}{T} \leq_e \frac{A}{T}$ and $\frac{A \cap B}{T} \leq_e \frac{B}{T}$, by [5, Lem. 2.3, p. 17]. Thus $\frac{A}{T}\ \beta\ \frac{B}{T}$.
⇐) Let $\frac{A}{T}\ \beta\ \frac{B}{T}$, then $\frac{A \cap B}{T} \leq_e \frac{A}{T}$ and $\frac{A \cap B}{T} \leq_e \frac{B}{T}$. Then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$, by [5, Lem. 2.3, P. 17]. Thus $A\ \beta_T\ B$.

Proposition (3.10): The β_T is an equivalence relation.

Proof: Clearly that β_T is reflexive and symmetric. We want to show β_T is transitive, let A, B and $C \in S_T$ such that $A\ \beta_T\ B$ and $B\ \beta_T\ C$. Since $A\ \beta_T\ B$ and $B\ \beta_T\ C$, then $A \cap B \leq_{Tes} A$, $A \cap B \leq_{Tes} B$, $B \cap C \leq_{Tes} B$ and $B \cap C \leq_{Tes} C$. Let L be a submodule of A such that $(A \cap C) \cap L \leq T$, then $(B \cap C) \cap (A \cap B \cap L) \leq T$. Since $B \cap C \leq_{Tes} B$, then $A \cap B \cap L \leq T$. Hence $(A \cap B) \cap (A \cap L) \leq T$. But $A \cap B \leq_{Tes} A$, therefore $A \cap L \leq T$. Since $L \leq A$, then $L \leq T$. So$A \cap C \leq_{Tes} A$. Similarly $A \cap C \leq_{Tes} C$. Thus β_T is an equivalence relation.

Proposition (3.11): Let T, A and B be submodules of a module M such that A and $B \in S_T$. Then $A\ \beta_T\ B$ if and only if

Proof: ⇒) Suppose that $A\ \beta_T\ B$ and let C be a submodule of M such that $A \cap C \leq T$. Then $A \cap B \cap C \leq T$, hence $(A \cap B) \cap (B \cap C) \leq T$. But $A \cap B \leq_{Tes} B$, therefore $B \cap C \leq T$. Now let $B \cap D \leq T$, where D is a submodule of M. Then $A \cap B \cap D \leq T$ and hence $(A \cap B) \cap (A \cap D) \leq T$. But$A \cap B \leq_{Tes} A$, therefore $A \cap D \leq T$.
⇐) To show $A\ \beta_T\ B$. Let L be a submodule of A such that $A \cap B \cap L \leq T$. Since $A \cap (B \cap L) \leq T$, then by our assumption $B \cap L = B \cap (B \cap L) \leq T$. Hence $A \cap L \leq T$. But $L \leq A$, therefore $L \leq T$. Similarly, let K be a Submodule of B such that $(A \cap B) \cap K \leq T$. Then by our assumption $A \cap K$

$= A \cap (A \cap K) \leq T$. Hence $B \cap K \leq T$. But $K \leq B$, therefore $K \leq T$. Thus $A \ \beta_T \ B$.

Proposition (3.12): Let T, A and B be submodules of a module M such that A and $B \in S_T$. Then $A \ \beta_T \ B$ if and only if for each $x \in A - T$, $y \in B - T$ there exists $r , r_1 \in R$ such that $rx \in B - T$ and $r_1 y \in A - T$.

Proof: Assume that $A \ \beta_T \ B$, then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Hence for each $x \in A - T$ there exists $r \in R$ such that $rx \in (A \cap B) - T$. Thus $rx \in B - T$. Similarly , for each $y \in B - T$ there exists $r_1 \in R$ such that $r_1 y \in (A \cap B) - T$ and hence $r_1 y \in A - T$.

For the converse, assume that $x \in A - T$. Then there exists $r \in R$ such that $rx \in B - T$. So $rx \in (A \cap B) - T$. Thus $A \cap B \leq_{Tes} A$. Now let $y \in B - T$, then there exists $r_1 \in R$ such that $r_1 y \in A - T$. Hence $r_1 y \in (A \cap B) - T$. So $A \cap B \leq_{Tes} B$. Thus $A \ \beta_T \ B$.

Proposition (3.13): Let T , A_1 , A_2 , B_1 and B_2 be submodules of a module M such that A_1, A_2, B_1 and $B_2 \in S_T$. If $A_1 \ \beta_T \ B_1$ and $A_2 \ \beta_T \ B_2$, then $(A_1 \cap A_2) \ \beta_T \ (B_1 \cap B_2)$.

Proof: Suppose that $A_1 \ \beta_T \ B_1$ and $A_2 \ \beta_T \ B_2$. Then $A_1 \cap B_1 \leq_{Tes} A_1$, $A_1 \cap B_1 \leq_{Tes} B_1$, $A_2 \cap B_2 \leq_{Tes} A_2$ and $A_2 \cap B_2 \leq_{Tes} B_2$. Hence $(A_1 \cap A_2) \cap (B_1 \cap B_2) \leq_{Tes} A_1 \cap A_2$ and $(A_1 \cap A_2) \cap (B_1 \cap B_2) \leq_{Tes} B_1 \cap B_2$, by [9, Prop.2.6 . P. 903]. Hence $(A_1 \cap A_2) \ \beta_T \ (B_1 \cap B_2)$.

Proposition (3.14): Let $f : M \to N$ be an epimorphism and T , A , B be submodules of N such that A and $B \in S_T$. If $A \ \beta_T \ B$, then $f^{-1}(A) \ \beta_{f^{-1}(T)} \ f^{-1}(B)$.

Proof: Let A and B be submodules of N such that $A \ \beta_T \ B$, then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Hence by [5, Lem. 2.15, P. 20] , $f^{-1}(A \cap B) \leq_{f^{-1}(T)es} f^{-1}(A)$, implies that $f^{-1}(A) \cap f^{-1}(B) \leq_{f^{-1}(T)es} f^{-1}(A)$. Since $A \cap B \leq_{Tes} B$, then by [5, Lem. 2.15, P. 20], $f^{-1}(A \cap B) \leq_{f^{-1}(T)es} f^{-1}(B)$, implies that $f^{-1}(A) \cap f^{-1}(B) \leq_{f^{-1}(T)es} f^{-1}(B)$. Thus $f^{-1}(A) \ \beta_T \ f^{-1}(B)$.

Proposition (3.15): Let T , A and B be submodules of a module M such that A and $B \in S_T$. If $A \ \beta_T \ B$, then $\frac{A}{A \cap B}$ and $\frac{B}{A \cap B}$ are singular.

Proof: Assume that $A \ \beta_T \ B$. Then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Then $\frac{A \cap B}{T} \leq_e \frac{A}{T}$ and $\frac{A \cap B}{T} \leq_e \frac{B}{T}$, by [5, Lem. 2.3, P. 17].

Now consider the following two short exact sequences:

$$0 \to \frac{A \cap B}{T} \xrightarrow{i} \frac{A}{T} \xrightarrow{\pi_1} \frac{A/T}{(A \cap B)/T} \to 0$$

$$0 \to \frac{A \cap B}{T} \xrightarrow{j} \frac{B}{T} \xrightarrow{\pi_2} \frac{B/T}{(A \cap B)/T} \to 0$$

where i , j are inclusion map and π_1 , π_2 are the natural epimorphisms . Since $\frac{A \cap B}{T} \leq_e \frac{A}{T}$ and $\frac{A \cap B}{T} \leq_e \frac{B}{T}$, then $\frac{A/T}{(A \cap B)/T}$ and $\frac{B/T}{(A \cap B)/T}$ are singular , by [2, Prop.1.20, P.31] . Hence by the third isomorphism theorem, $\frac{A/T}{(A \cap B)/T} \cong \frac{A}{A \cap B}$ and $\frac{B/T}{(A \cap B)/T} \cong \frac{A}{A \cap B}$. Thus $\frac{A}{A \cap B}$ and $\frac{B}{A \cap B}$ are singular.

Corollary (3.16): Let T , A and B be submodules of a module M such that A and $B \in S_T$. If $A \ \beta_T \ B$, then $\frac{A+B}{A}$ and $\frac{A+B}{B}$ are singular.

Proof: Clear by the second isomorphism theorem.

Proposition (3.17): Let $\{M_\alpha\}_{\alpha \in \Lambda}$ be a family of modules and T_α , A_α and B_α be submodules of M_α , for each $\alpha \in \Lambda$ such that $T_\alpha \leq A_\alpha \cap B_\alpha$. If $A_\alpha \ \beta_{T_\alpha} \ B_\alpha$, for each $\alpha \in \Lambda$, then $(\oplus_{\alpha \in \Lambda} A_\alpha) \ \beta_{\oplus_{\alpha \in \Lambda} T_\alpha} \ (\oplus_{\alpha \in \Lambda} B_\alpha)$.

Proof : Let $A_\alpha \ \beta_{T_\alpha} \ B_\alpha$ for each $\alpha \in \Lambda$,then $A_\alpha \cap B_\alpha \leq_{(T_\alpha)es} A_\alpha$ and $A_\alpha \cap B_\alpha \leq_{(T_\alpha)es} B_\alpha$. Hence by [5, Lem. 2.3, P. 17] . $\frac{A_\alpha \cap B_\alpha}{T_\alpha} \leq_e \frac{A_\alpha}{T_\alpha}$ and $\frac{A_\alpha \cap B_\alpha}{T_\alpha} \leq_e \frac{B_\alpha}{T_\alpha}$. Then by [1], $\oplus_{\alpha \in \Lambda} \frac{A_\alpha \cap B_\alpha}{T_\alpha} \leq_e \oplus_{\alpha \in \Lambda} \frac{A_\alpha}{T_\alpha}$ and $\oplus_{\alpha \in \Lambda} \frac{A_\alpha \cap B_\alpha}{T_\alpha} \leq_e \oplus_{\alpha \in \Lambda} \frac{B_\alpha}{T_\alpha}$. But $\oplus_{\alpha \in \Lambda} \frac{A_\alpha \cap B_\alpha}{T_\alpha} \cong \frac{\oplus_{\alpha \in \Lambda}(A_\alpha \cap B_\alpha)}{\oplus_{\alpha \in \Lambda} T_\alpha}$, $\oplus_{\alpha \in \Lambda} \frac{A_\alpha}{T_\alpha} \cong \frac{\oplus_{\alpha \in \Lambda} A_\alpha}{\oplus_{\alpha \in \Lambda} T_\alpha}$ and $\oplus_{\alpha \in \Lambda} \frac{B_\alpha}{T_\alpha} \cong \frac{\oplus_{\alpha \in \Lambda} B_\alpha}{\oplus_{\alpha \in \Lambda} T_\alpha}$, therefore $\frac{\oplus_{\alpha \in \Lambda}(A_\alpha \cap B_\alpha)}{\oplus_{\alpha \in \Lambda} T_\alpha} \leq_e \frac{\oplus_{\alpha \in \Lambda} A_\alpha}{\oplus_{\alpha \in \Lambda} T_\alpha}$ and $\frac{\oplus_{\alpha \in \Lambda}(A_\alpha \cap B_\alpha)}{\oplus_{\alpha \in \Lambda} T_\alpha} \leq_e \frac{\oplus_{\alpha \in \Lambda} B_\alpha}{\oplus_{\alpha \in \Lambda} T_\alpha}$. Then by [5, Lem. 2.3, P. 17], $\oplus_{\alpha \in \Lambda} (A_\alpha \cap B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T_\alpha)es} \oplus_{\alpha \in \Lambda} A_\alpha$ and hence $\oplus_{\alpha \in \Lambda} (A_\alpha \cap B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T_\alpha)es} \oplus_{\alpha \in \Lambda} B_\alpha$. Hence $(\oplus_{\alpha \in \Lambda} A_\alpha) \cap (\oplus_{\alpha \in \Lambda} B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T_\alpha)es} \oplus_{\alpha \in \Lambda} A_\alpha$ and $(\oplus_{\alpha \in \Lambda} A_\alpha) \cap (\oplus_{\alpha \in \Lambda} B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T_\alpha)es} \oplus_{\alpha \in \Lambda} B_\alpha$. Thus $(\oplus_{\alpha \in \Lambda} A_\alpha) \ \beta_{\oplus_{\alpha \in \Lambda} T_\alpha} \ (\oplus_{\alpha \in \Lambda} B_\alpha)$.

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