Local Fusion Graphs of Double Covers of Certain Mathieu Groups and Their Automorphism Groups

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***Abstract*:** Let G a finite group and X a subset of G. The local fusion graph denoted by Ƒ(G,X) has a vertex set X with two distinct element $x \neq y \in X$ are adjacent if the group generated by x and y, $\langle x,y \rangle$, is dihedral group, of order 2n, n odd. In this paper we prove that the local fusion graphs for Mathieu groups and their Automorphism groups has diameter 2.

***Keywords*:** Double covers of Mathieu groups, Local Fusion Graphs, Collapsed Adjacency Matrices, Diameters.

1. Introduction

Recently, the study of the action of the group on graph has been shown to be effective when studying properties of a group. Suppose that G a group with finite order and X class of involution in G, the local fusion graph denoted by Ƒ(G,X) has a vertex set X with two distinct vertices are connected if the group generated by x and y, $\langle x,y \rangle$, is dihedral group of order 2n, n odd, so x conjugate to y in $\langle x,y \rangle$. Studying the structure of groups by using the associated local fusion graphs can be seen in [1]–[3] where X taken to be a conjugacy class of involution. This paper deal with local fusion graphs computationally , the computer algebra systems Magma [4] and GAP [5] have been employed for this purpose. Also, the group representation which define in Magma and GAP can be obtained from the online Atlas of Group Representations [6]. One can show immediately that G induces graph automorphisms on the local fusion graph Ƒ(G,X) (by conjugation) and acts transitively on the graph vertices. For distinct x, $y \in X$, a distance between x and y, d(x,y), is a shortest path between x and y. Also the i^{th} disc of the element $x \in X$, $\Delta_i(x)$, is the set of vertices of Ƒ(G,X) which has distance i from x, also, we may let Diam(Ƒ(G,X)) to be the diameter of Ƒ(G,X) . Let $x \in$ G the Centralizer (the set of elements in G commute with x) in G of x G_x (= $C_G(x)$). Clearly , $\Delta_i(x)$ equal a union of certain $C_G(x)$ -orbits. Finally, we should mentioned that the notations of this groups from Atlas [7]. The main goal of this paper is to investigate the local fusion graphs for Mathieu groups and their Automorphism and we prove computationally both graphs have diameter 2.

2. Main Results

In the 19th century Emile Mathieu discovered the Mathieu groups which are the first family of sporadic simple groups (see [8], [9]).

In this paper we study the local fusion graph for the following groups:

- $2.M_{12}.2$ for the class 2D (class of elements of order 2) with size 1584.
- $2.M_{22}.2$ for the class 2F (class of elements of order 2) with size 2772.

As the rest of the classes divide into different classes with isomorphic local fusion graph see [10]. Let t be a fixed involution (element of order 2) in either 2D or 2F. Since the center of the above groups is cyclic group of order 2, generated by involution say ς, then by [6] one can see that $t\varsigma \in t^G$. Thus for any involution $\eta \in t^G$, the element $t\varsigma$ has even order in G. For that reason we assume that $X=2D \backslash \{t\varsigma\}$ or $X=2F \backslash \{t\varsigma\}$. Magma can provide a code to find the permutation rank of $C_G(t)$ on 2D or 2F which is equal to the number of $C_G(t)$-orbits under the action on 2D or 2F by conjugation, and this for the case 2D and 2F is 27 and 28, respectively.

Let C be a Conjugate class in G so that($C=\{xcx^{-1} | x \in G, c \in C\}=c^G$) , then the set X_C defined to be the set of all element $x \in X$, such that $tx \in X$. Obviously, $C_G(t)$ breaks up into suborbits by its action on X_C, C all over the classes of G. And by [11] the following formula gives us the size of the set X_C

$$|X_C| = \left(\frac{|G|}{|C_{G(g)}||C_{G(t)}|}\right)\left(\sum_{\chi \in Irr} \frac{\chi(g)\chi(t)\chi(t)}{\chi(1)}\right) \qquad (1)$$

Where the sum is over all of the irreducible characters $\chi(g)$ of G, for $g \in G$. The previous formula is calculated by GAP using the code **"Class Multiplication Coefficient"**. Thus the size of X_C now available computationally.

Now we explain a procedure to find the $\Delta_i(t) \cap X_C$ for the above groups. In order to do that we define the following algorithm which aim to find suborbit representatives. The structure of this algorithm summarized as follows :

Algorithm 1.

Input: G is either $2.M_{12}.2$ or $2.M_{22}.2$, t involution in 2D or 2F,respectively;
i: r $\longrightarrow$ Random($t^G \setminus \{t\}$)
ii: set Reps $\longrightarrow$ {r} and CR $\longrightarrow$ r^{Gt} .
iii: **for** $x \in t^G \setminus \{t\}$ check **if** $x \notin$ CR (symbol for r^{Gt}), **then**
iv: CR $\longrightarrow$ CR $\cup$ $\{x^{Gt}\}$; and Reps $\longrightarrow$ Reps $\cup$ {x}.
Output: The set of suborbit representatives.

The next result cope with the diameters of the local fusion graphs Ƒ($2.M_{12}.2$, 2D\{ tς}) and Ƒ($2.M_{22}.2$, 2F\{ tς}).

Theorem 1. The Diameter of local fusion graphs Ƒ($2.M_{12}.2$, 2D\{ tς}) and Ƒ($2.M_{22}.2$, 2F\{ tς}) equal 2

Proof: We have form the output of **Algorithm 1** we find 27 and 28 representatives for $C_G(t)$-orbits for the graphs Ƒ($2.M_{12}.2$, 2D\{ tς}) and Ƒ($2.M_{22}.2$, 2F\{ tς}). Furthermore, the Magma code **"Is Conjugate"** is in service to find the set of conjugacy classes such that $X_C \neq \phi$.
From that we can get the G-classes such that X_C is non-empty for both graphs:
{2ABC,3AB,4A,5B,6ABCD,10A,11A,12A,20A,22}
and {2ADE,3A,4CDF,5A,6ABC,10A,11A,22A},
respectively.
The graph Ƒ($2.M_{12}.2$, 2D\{ tς}) has 16 class make $X_C \neq \phi$. Obviously, $X_{\{3AB,5A,11A\}}$ in the $\Delta_1(t)$ and the reminder classes cannot be in $\Delta_1(t)$ this because they have even order if we multiply their representative by t. Now to check the reminder classes lie in $\Delta_2(t)$ we first find the whole $\Delta_1(t)$ and then search for a random element $y \in \Delta_1(t)$ and we see that there is an element z in X_C such that C is even class with property <y,z> is dihedral group of order 2n, n odd.
Thus:

$$\text{Diam }(Ƒ(2.M_{12}.2,\ 2D\backslash\{ tς\})) = 2.$$

Similar approach could be utilized to prove that:

$$\text{Diam }(Ƒ(2.M_{22}.2,\ 2F\backslash\{ tς\})) = 2$$ □

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The proof of Theorem 1 computationally can be explained as follows:

1. Use the magma code **"Is Conjugate"** break up the set X_C into the non-empty classes.
2. $\Delta_1(t)$ representative is the one in X_C such that C is odd call this set of representative by SubRep. Then
3. $\Delta_1(t) = \bigcup_{x \in SubRep} Conugate(C_G(t), x)$.
4. The reminder class named by RemSubRep
5. For y in $-\Delta_1(t)$ there is an element x in RemSubRep such that yx has odd order.
6. $\Delta_2(t) = \bigcup_{x \in RemSubRep} Conugate(C_G(t), x)$.

The structure of the local fusion graphs Ƒ($2.M_{12}.2$, 2D\{ tς}) and Ƒ($2.M_{22}.2$, 2F\{ tς}) are described in the next result.

Theorem 2 The discs structural of local fusion graphs Ƒ(2.M12.2, 2D\{ tς}) and Ƒ($2.M_{22}.2$, 2F\{ tς}) can be explain in the following tables:

Table1 Ƒ($2.M_{12}.2$, 2D\{ tς})

X_C G-conjugacy Classes	$\Delta_1(t)$	$\Delta_2(t)$
3A	20,20	
3B	60	
5A	60,60	
11A	120,120	
2BC		15
4A		30,2
6A		20,20
6B		20
6CD		60,60
10A		60,60
12A		120
20A		120,120
22A		120,120

Table 2 Ƒ($2.M_{22}.2$, 2F\{ tς})

X_C G-conjugacy Classes	$\Delta_1(t)$	$\Delta_2(t)$
3A	40,40	
5A	160,160	
11A	320,320	
2DE		5,20
4CD		80
4F		40,40,40,40
6A		40,40
6BC		80,80
10A		160,160
22A		320,320

Proof: Theorem 1 shows that the diameters for both graphs are equal 2. Also, the Gap code "**Class Multiplication Coefficien**t" may apply to find the sizes of X_C, which break up to suborbits. To calculate the size of arbitrary suborbits say $x \in X_C$ we divide the $|C_G(t)|$ by $|C_{C_G(t)}(x)|$ which can be done by using the magma code
"**Order(Centraliser(G,t)/ Centraliser(Centraliser(G,t),x))**". □

3. The Collapsed Adjacency Matrices

For a given two $C_G(t)$-orbits, say o_i, o_j the collapsed adjacency matrix for the local fusion graph Ƒ(G,X) has entry, represent the number of the edges in the orbit o_j that are connected to a single vertex in the orbit o_i. In the following matrices we change each orbit in **Table 1** and **Table 2** with $\Delta_i^j(t)$ Increasingly, also we let $\Delta_0^1(t)$=t . The next tables 3,4 presents the collapsed adjacency matrix for the local fusion graph Ƒ(G,X), such that **Table 3** gives the details for the graph Ƒ(2.M_{12}.2, 2D\{ tς}), whereas **Table 4** provides the information for the graph Ƒ(2.M_{22}.2, 2F\{ tς}):

Table3: The Collapsed Adjacency Matrices for Ƒ(2.M_{12}.2, 2D\{ tς})

Class	Δ_0^1	Δ_1^1	Δ_1^2	Δ_1^3	Δ_1^4	Δ_1^5	Δ_1^6	Δ_1^7	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_2^9	Δ_2^{10}	Δ_2^{11}	Δ_2^{12}	Δ_2^{13}	Δ_2^{14}	Δ_2^{15}	Δ_2^{16}	Δ_2^{17}	Δ_2^{18}
Δ_0^1	0	20	20	60	60	60	120	120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	20	17	0	18	30	24	60	18	9	0	1	9	10	0	18	24	12	30	6	6	12	36	36	36	18	30
Δ_1^2	20	0	17	18	30	24	18	60	9	0	1	9	0	10	18	12	24	6	30	6	12	36	36	36	30	18
Δ_1^3	60	6	6	20	15	19	30	30	0	6	0	8	6	6	24	14	14	20	20	24	12	36	36	28	40	40
Δ_1^4	60	10	10	15	24	23	42	42	8	6	0	8	2	2	24	18	18	14	14	8	16	32	36	36	26	26
Δ_1^5	60	8	8	19	23	28	36	36	4	2	0	8	4	4	12	16	16	18	18	16	16	36	28	36	34	34
Δ_1^6	12 0	10	3	15	21	18	40	28	5	4	1	9	5	3	20	17	17	24	18	13	17	34	35	35	28	40
Δ_1^7	12 0	3	10	15	21	18	28	40	5	4	1	9	3	5	20	17	17	18	24	13	17	34	35	35	40	28
Δ_2^1	15	12	12	0	32	16	40	40	13	0	0	0	0	0	24	12	12	28	28	24	8	32	32	32	32	32
Δ_2^2	15	0	0	24	24	8	32	32	0	13	0	0	12	12	0	28	28	12	12	32	16	32	32	32	40	40
Δ_2^3	30	10	10	0	0	0	60	60	0	0	1	0	10	10	0	0	0	0	0	0	0	60	60	60	60	60
Δ_2^4	2	6	6	16	16	16	36	36	0	0	0	13	6	6	16	20	20	20	20	16	16	56	24	24	36	36
Δ_2^5	20	10	0	18	6	12	30	18	0	9	1	9	17	0	18	30	6	24	12	30	24	36	36	36	18	60
Δ_2^6	20	0	10	18	6	12	18	30	0	9	1	9	0	17	18	6	30	12	24	30	24	36	36	36	60	18
Δ_2^7	20	6	6	24	24	12	40	40	6	0	0	8	6	6	20	20	20	14	14	15	19	36	36	28	30	30
Δ_2^8	0	8	4	14	18	16	34	34	3	7	0	10	10	2	20	19	16	18	12	14	18	32	34	34	36	48
Δ_2^9	0	4	8	14	18	16	34	34	3	7	0	10	2	10	20	16	19	12	18	14	18	32	34	34	48	36
Δ_2^{10}	0	10	2	20	14	18	48	36	7	3	0	10	8	4	14	18	12	19	16	18	16	32	34	34	34	34
Δ_2^{11}	0	2	10	20	14	18	36	48	7	3	0	10	4	8	14	12	18	16	19	18	16	32	34	34	34	34
Δ_2^{12}	0	2	2	24	8	16	26	26	6	8	0	8	10	10	15	14	14	18	18	24	23	32	36	36	42	42
Δ_2^{13}	0	4	4	12	16	16	34	34	2	4	0	8	8	8	19	18	18	16	16	23	28	36	28	36	36	36
Δ_2^{14}	0	6	6	18	16	18	34	34	4	4	1	14	6	6	18	16	16	16	16	16	18	32	39	39	34	34
Δ_2^{15}	0	6	6	18	18	14	35	35	4	4	1	6	6	6	18	17	17	17	17	18	14	39	44	31	35	35
Δ_2^{16}	0	6	6	14	18	18	35	35	4	4	1	6	6	6	14	17	17	17	17	18	18	39	31	44	35	35
Δ_2^{17}	0	3	5	20	13	17	28	40	4	5	1	9	3	10	15	18	24	17	17	21	18	34	35	35	40	28
Δ_2^{18}	0	5	3	20	13	17	40	28	4	5	1	9	10	3	15	24	18	17	17	21	18	34	35	35	28	40

Table 4 : The Collapsed Adjacency Matrices for $\mathcal{F}(2.M_{22}.2, 2F\backslash\{ t\varsigma\})$

Class	Δ_0^1	Δ_1^1	Δ_1^2	Δ_1^3	Δ_1^4	Δ_1^5	Δ_1^6	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_2^9	Δ_2^{10}	Δ_2^{11}	Δ_2^{12}	Δ_2^{13}	Δ_2^{14}	Δ_2^{15}	Δ_2^{16}	Δ_2^{17}	Δ_2^{18}	Δ_2^{19}	Δ_2^{20}
Δ_0^1	0	40	40	160	160	320	320	5	20	5	20	80	80	40	40	40	40	40	40	80	80	80	80	160	160	320	320
Δ_1^1	40	25	0	56	80	112	136	0	2	12	1	36	32	0	20	16	12	32	0	32	36	32	32	40	72	80	144
Δ_1^2	40	0	25	80	56	136	112	0	2	12	1	36	32	20	0	12	16	0	32	36	32	32	32	72	40	144	80
Δ_1^3	160	14	20	77	68	130	122	4	3	10	2	28	38	16	12	14	16	10	18	30	30	34	36	52	40	114	102
Δ_1^4	160	20	14	68	77	122	130	4	3	10	2	28	38	12	16	16	14	18	10	30	30	36	34	40	52	102	114
Δ_1^5	320	14	17	65	61	125	104	6	3	9	2	27	38	19	12	13	16	10	18	29	31	35	36	57	51	126	116
Δ_1^6	320	17	14	61	65	104	125	6	3	9	2	27	38	12	19	16	13	18	10	31	29	36	35	51	57	116	126
Δ_2^1	5	0	0	32	32	96	96	17	0	0	0	64	0	16	16	16	16	24	24	52	52	20	20	80	80	144	144
Δ_2^2	20	16	16	96	96	192	192	0	1	0	0	0	0	16	16	0	0	8	8	0	0	0	0	64	64	128	128
Δ_2^3	5	24	24	80	80	144	144	0	0	17	0	0	64	16	16	16	16	0	0	20	20	52	52	32	32	96	96
Δ_2^4	20	8	8	64	64	128	128	0	0	0	1	0	0	16	16	0	0	16	16	0	0	0	0	96	96	192	192
Δ_2^5	80	18	18	56	56	108	108	16	0	0	0	21	16	16	16	18	18	16	16	18	18	16	16	76	76	152	152
Δ_2^6	80	16	16	76	76	152	152	0	0	16	0	16	21	16	16	18	18	18	18	16	16	18	18	56	56	108	108
Δ_2^7	40	0	20	64	48	152	96	8	2	8	2	32	32	37	0	12	12	0	20	32	36	36	32	64	48	152	96
Δ_2^8	40	20	0	48	64	96	152	8	2	8	2	32	32	0	37	12	12	20	0	36	32	32	36	48	64	96	152
Δ_2^9	40	16	12	56	64	104	128	8	0	8	0	36	36	12	12	17	16	12	16	36	32	36	32	64	56	128	104
Δ_2^{10}	40	12	16	64	56	128	104	8	0	8	0	36	36	12	12	16	17	16	12	32	36	32	36	56	64	104	128
Δ_2^{11}	40	32	0	40	72	80	144	12	1	0	2	32	36	0	20	12	16	25	0	32	32	36	32	56	80	112	136
Δ_2^{12}	40	0	32	72	40	144	80	12	1	0	2	32	36	20	0	16	12	0	25	32	32	32	36	80	56	136	112
Δ_2^{13}	80	16	18	60	60	116	124	13	0	5	0	18	16	16	18	18	16	16	16	21	16	16	18	72	68	144	140
Δ_2^{14}	80	18	16	60	60	124	116	13	0	5	0	18	16	18	16	16	18	16	16	16	21	18	16	68	72	140	144
Δ_2^{15}	80	16	16	68	72	140	144	5	0	13	0	16	18	18	16	18	16	18	16	16	18	21	16	60	60	124	116
Δ_2^{16}	80	16	16	72	68	144	140	5	0	13	0	16	18	16	18	16	18	16	18	18	16	16	21	60	60	116	124
Δ_2^{17}	160	10	18	52	40	114	102	10	2	4	3	38	28	16	12	16	14	14	20	36	34	30	30	77	68	130	122
Δ_2^{18}	160	18	10	40	52	102	114	10	2	4	3	38	28	12	16	14	16	20	14	34	36	30	30	68	77	122	130
Δ_2^{19}	320	10	18	57	51	126	116	9	2	6	3	38	27	19	12	16	13	14	17	36	35	31	29	65	61	125	104
Δ_2^{20}	320	18	10	51	57	116	126	9	2	6	3	38	27	12	19	13	16	17	14	35	36	29	31	61	65	104	125

Conclusion: A good deal of researches have been achieved during this paper. For example, the disc structure of certain local fusion graphs were determined. Moreover, calculating the diameters of these graphs is the most notable of what has been achieved . Also, the collapsed adjacency matrix for the local fusion graphs has been accomplished. Finally, computational approaches were most applied for analyzing the aforementioned local fusion graphs.

References:

[1] A. Devillers and M.Giudici, "Involution Graphs Where The Product Of Two Adjacent Vertices Has Order Three," *J. Aust. Math. Soc.*, vol. 85, no. 3, p. 305, Dec. 2008.

[2] J. Ballantyne, N. Greer, and P. Rowley, "Local fusion graphs for symmetric groups," *J. Gr. Theory*, vol. 16, no. 1, pp. 35–49, Jan. 2013.

[3] J. Ballantyne, "On local fusion graphs of finite Coxeter groups," *J. Gr. Theory*, vol. 16, no. 4, pp. 595–617, Jan. 2013.

[4] W. Bosma, J. Cannon, and C. Playoust, “The Magma Algebra System I: The User Language,” *J. Symb. Comput.*, vol. 24, no. 3–4, pp. 235–265, Sep. 1997.

[5] The GAP Group, “GAP Groups, Algorithms, and Programming, ”. *http://www.gap-system.org*. Version 4.4.12, 2008.

[6] R. A. Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nickerson, Steve Linton, John Bray, “ATLAS of Finite Group Representations - V3.” [Online]. Available: http://brauer.maths.qmul.ac.uk/Atlas/v3/. Dec, 2018.

[7] J. H. Conway, Atlas of finite groups: maximal subgroups and ordinary characters for simple groups, *1st ed*. Oxford University Press, 1985.

[8] E. L. Mathieu, “Memoire sur l’ et ude des fonctions de plusieurs quantites, sur la maniere des forms et sur les substitutions qui les laissent invariables,” *J. Math. Pures Appl.*, vol. 6, pp. 241–323, 1861.

[9] É. L. Mathieu, “Sur la fonction cinq fois transitive de 24 quantités.,” *J. Math. Pures Appl.*, pp. 25–46, 1873.

[10] D. Gorenstein, *Finite Groups*, 2nd ed. Chelsea Publishing Company, 1980.

[11] J. F. Humphreys, “Conjugacy classes of double covers of monomial groups,” *Math. Proc. Cambridge Philos. Soc.*, vol. 96, no. 2, p. 195, Sep. 1984.