ON A CLASS OF ANALYTIC MULTIVALENT FUNCTIONS INVOLVING HIGHER-ORDER DERIVATIVES

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Abstract

In this paper, we get some interesting geometric concepts of the class of multivalent functions involving higher order derivatives defined on the open unite disk U. We obtain some interesting properties, like , coefficient inequalities, distortion and growth property , closure property, radius of stalikness and radius of convexity and hadamard product .

Keywords: Analytic , Multivalent , higher order derivatives.

1.Introduction.

Let $S_p(n)$ denoted the class of analytic functions:
$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $(p, n \in \mathbb{N} = \{1, 2, 3, \dots\})$ ... (1.1), are p−valent in unit disc U = { {z : z ∈ ℂ and |z| < 1}. Let $T_p(n)$ denoted the subclass of Sp(n) of the following form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0) \dots\dots (1.2)$$

We note that $T_p(1) = T_p$.

For all $(z) \in S_p(n)$, we have
$f^{(m)}(z) = \delta(p, m) z^{p-m} + \sum_{k=p+n}^{\infty} \delta(k, m) a_k z^{k-m}$ (1.3)

Where

$$\delta(i, j) = \frac{i!}{(i-j)!}$$

$$= \begin{cases} i(i-1)(i-2)\dots\dots(i-j+1) & j \neq 0 \\ 1 & j = 0 \end{cases} \dots\dots (1.4)$$

Aouf [1] introduced and studied the class $T_p^*(\lambda, l, \alpha, \beta)$ consisting of functions $f(z) \in S_p(n)$ which satisfies:

$$\left| \frac{A\left\{\frac{f'(z)}{z^{p-1}} - p(p-1)\right\}}{B\left\{\frac{f'(z)}{z^{p-1}} - p(p-1)\right\} + \lambda(1-\alpha)} \right| < (l-\beta) \dots\dots\dots (1.5)$$

Where $0 < B \leq 1$, $A \geq 0$, $\lambda > 0$, $0 \leq \alpha < 1$, $0 < l < B < 1$, $p \in N$ and $z \in U$.

Let S_n $(p, q;, A, B, \lambda, \alpha, l, \beta)$be the subclass of $S_p(n)$ consisting of functions $f(z)$ of the form (1.1), and satisfying the analytic criterion:

$$\left| \frac{A\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q) z^{p-q-2}} - p(p-1)\right\}}{B\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q) z^{p-q-2}} - p(p-1)\right\} + \lambda(1-\alpha)} \right| < (l-\beta) \dots\dots\dots (1.6)$$

Where $0 < B \leq 1$, $A \geq 0$, $\lambda > 0$, $0 \leq \alpha < 1$, $0 < l < B < 1$, $p \in N_0 = N \cup \{0\}$ and $p > q$

Further, let
T_n^* $(p, q; A, B, \lambda, \alpha, l, \beta) = S^*$ $(p, q;$ $A, B, \lambda, \alpha, l, \beta) \cap T_p(n)$ .... (1.7)

For suitable choices of $n, p, q, A, B, \lambda, l, \alpha$ and β we obtain the following subclasses:

(i) $T_1^*(p, 0;, A, B, \lambda, \alpha, l, \beta) = P_p^*(A, B, \lambda, \alpha, l, \beta)$ $(Aouf[1])$;

(ii) $T_1^*(1, 0; A, B, \lambda, \alpha, l, \beta) = P^*(A, B, \lambda, \alpha, l, \beta)$ $(Gupta$ and $Jain[2])$

(iii) $T_1^*(p, 0;, A, B, \lambda, \alpha, l, 1) = F_p(A, B, \lambda, 1, l, \alpha)$ $(Lee$ et $al[3])$

Also , we note that :
T_n^* $(p, q; A, B, \lambda, \alpha, l, 1) = T_n^*(p, q; A, B, \lambda, \alpha, l) = \left\{ f \in T_p(n) : Re\left(\frac{f^{(q+2)}(z)}{\delta(p-2,q) z^{p-q-2}}\right) > \alpha, 0 \leq \alpha < p \right\}$

2.Coefficient inequalities

We assume throughout this paper that $0 < B \leq 1$, $A \geq 0$, $\lambda > 0$, $0 \leq \alpha < 1$, $0 < l < B < 1$, $p \in N_0 = N \cup \{0\}$ and $p > q$ and $\delta(i, j)(i > j)$ is defined by (1.4).

theorem 1. A function $f(z)$ of the form (1.2) is in the class T_n^* $(p, q, A, B, \lambda, \alpha, l, \beta)$ if and only if
$\sum_{k=p+n}^{\infty} [A + B(l-\beta)] k(k-1) \delta(k-2, q) a_k \leq \lambda(l-\beta)(1-\alpha)\delta(p-2, q)$…..(2.1)

Proof. Assume that the inequality (2.1) holds true , then
$\left|A\left\{f^{(q+2)}(z) - p(p-1)\delta(p-2, q)\, z^{p-q-2}\right\}\right| - (l-\beta)\left|B\left\{f^{(q+2)}(z) - p(p-1)\, \delta(p-2, q) z^{p-q-2}\right\} + \lambda(1-\alpha)\delta(p-2, q)\, z^{p-q-2}\right|$

$=\left|A\left\{\delta(p,q+2)\, z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_k\, z^{k-q-2}-\ p(p-1)\,\delta(p-2,q)\, z^{p-q-2}\right\}\right|-(l-\beta)\left|B\left\{\delta(p,q+\ 2)z^{p-q-2}+\ \sum_{k=p+n}^{\infty}\delta(k,q+2)\, a_k\, z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right|$

We have $\delta(p,q+2)=p(p-1)\delta(p-2,q)$ then

$=\left|A\left\{p(p-1)\delta(p-2,q)z^{p-q-2}+\sum_{k=p+n}^{\infty}k(k-1,q)\delta(p-2,q)a_k z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}\right|-(l-\beta)\left|B\left\{p(p-1)\delta(p-2,q)z^{p-q-2}+\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right|$

then

$=\left|A\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}\right|-(l-\beta)\left|B\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right|$

$$\le\sum_{k=p+n}^{\infty}[A+B(l-\beta)](k(k-1)\delta(k-2,q)a_k|z|^{k-q-2}-\lambda(1-\alpha)(l-\beta)\delta(p-2,q)|z|^{k-q-2}$$

$=\sum_{k=p+n}^{\infty}[A+B(l-\beta)](k(k-1)\delta(k-2,q)a_k\le\lambda(1-\alpha)(l-\beta)\delta(p-2,q)$

Conversely , assume that $f(z)\in T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ thus

$$\left|\frac{A\left[f^{(q+2)}(z)-p(p-1)\delta(p-2,q)z^{p-q-2}\right]}{B\left\{f^{(q+2)}(z)-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}}\right|=$$

$$\left|\frac{A\left[\delta(p,q+2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_k z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right]}{B\left[\delta(p,q+2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_k z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right]+\lambda(1-\alpha)\delta(p-2,q)z^{p-q-2}}\right|<l-\beta\ .$$

We have $\delta(p,q+2)=p(p-1)\delta(p-2,q)$ then

$$\left|\frac{A\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}}{B\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}}\right|<l-\beta$$

Since $Re(z)\le|z|$ for all z , we get

$$Re\left|\frac{A\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}}{B\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_k z^{k-q-2}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}}\right|<l-\beta\ \ldots\ldots\ldots(2.2)$$

Taking values of z on the real axis then $\frac{f^{(q+2)}(z)}{\delta(p-2,q)z^{p-q-2}}$ is real then , upon cleaning the denominator in (2.2) and putting $z\to-1$, we get the desired result .

Corollary 1. Let the function $f(z)$defined by (1.1) be in the class $T_n^*\ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then

$$a_k\le\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}$$

Sharpness is hold for

$$f(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k,$$

$(k\ge n+p\,,n\in\mathcal{N})$

3.Distortion property

theorem 2. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then $|z|=r<1$ we have

$$\left(\delta(\mathrm{p,m})-\delta(\mathrm{p}+\mathrm{n,m})\frac{\lambda(\mathrm{l}-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(\mathrm{l}-\beta)](\mathrm{p}+\mathrm{n})(\mathrm{p}+\mathrm{n}-1)\delta(\mathrm{p}+\mathrm{n}-2,\mathrm{q})}\mathrm{r}^{\mathrm{m}}\right)\mathrm{r}^{\mathrm{p-m}}\le\left|\mathrm{f}^{(\mathrm{m})}(\mathrm{z})\right|\le\left\{\delta(\mathrm{p,m})+\delta(\mathrm{p}+\mathrm{n,m})\frac{\lambda(\mathrm{l}-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(\mathrm{l}-\beta)](\mathrm{p}+\mathrm{n})(\mathrm{p}+\mathrm{n}-1)\delta(\mathrm{p}+\mathrm{n}-2,\mathrm{q})}\mathrm{r}^{\mathrm{m}}\right\}\mathrm{r}^{\mathrm{p-m}}\ \ldots..(3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}z^{p+n}\ \ldots\ldots(3.2)$$

Proof . By theorem 1 , we have

$$[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)\sum_{k=p+n}^{\infty}a_k\le\lambda(l-\beta)(1-\alpha)\delta(p-2,q)$$

$$\le\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)a_k\le\lambda(l-\beta)(1-\alpha)\delta(p-2,q)\ \ldots\ldots(3.3)$$

That is

$$\sum_{k=p+n}^{\infty}a_k\le\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ \ldots\ldots(3.4)$$

From (1.3) and (3.4)

$$\left|f^{(m)}(z)\right|\ge\left\{\delta(p,m)r^{p-m}-r^{p+n-m}\delta(p+n,m)\sum_{k=p+n}^{\infty}a_k\right\}$$

$$\ge\left\{\delta(p,m)r^{p-m}-r^{p+n-m}\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}\right\}$$

$$\ge\left\{\delta(p,m)-\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}r^n\right\}r^{p-m}.\ .\ .\ .\ .\ .\ .\ (3.5)$$

and

$$\left|f^{(m)}(z)\right|\le\left\{\delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\sum_{k=p+n}^{\infty}a_k\right\}$$

$$\le\left\{\delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q))}\right\}$$

$$\le\left\{\delta(p,m)+\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}r^n\right\}r^{p-m}\ \ldots\ldots(3.6)$$

The proof of theorem 2 is done.

Putting m=0 in previous theorem 2 , we get the following corollary

Corollary 2. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then

$|z| = r < 1$

$$|f(z)| \geq \left\{1 - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^p \;\ldots\ldots (3.7)$$

and

$$|f(z)| \leq \left\{1 + \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^p \;\ldots\ldots (3.8)$$

The result is sharp.

Putting m = 1 in previous Theorem 2, we get

Corollary 3. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then $|z| = r < 1$ we have

$$|f'(z)| \geq \left\{p - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^{p-1} \;\ldots\ldots (3.9)$$

and

$$|f'(z)| \leq \left\{p + \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^{p-1} \;\ldots\ldots (3.10)$$

The result is sharp

Remark : Taking $q = 0$ *and* $n = 1$ in Corollaries 2 and 3 we obtain the result obtained by Aouf [3, theorem 2]

4.Radius of Starlikeness and Radius of convexity

Theorem 3. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then $f(z)$ is $p-$valent close to convexity of order η $(0 \leq \eta \leq p)$ in $|z| \leq r_1$ where

$$r_1 = inf\left\{\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)}\right\}^{\frac{1}{k-p}} \quad (k \geq n+p, p, n \in N) \;\ldots\ldots (4.1)$$

The result is sharp , the extremal function given by (2.4).

Proof : we must show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \eta \quad \text{for } |z| \leq r_1 \quad \ldots..(4.2)$$

where r_1 is given by (4.1). Indeed we find from (1.2) that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+n}^{\infty} k a_k |z|^{k-p}$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \eta$$

If

$$\sum_{k=p+n}^{\infty}\left(\frac{k}{p-\eta}\right) a_k |z|^{k-p} \leq 1 \;\ldots\ldots\ldots.(4.3)$$

By theorem 1 , (4.3) will be hold if

$$\left(\frac{k}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q2}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)$$

Then

$$|z| \leq \left(\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)^{\frac{1}{k-p}} \ldots\ldots\ldots\ldots(4.4)$$

The result is follow from (4.4)

Theorem 4. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$then $f(z)$ is $p-valent$ starlikeness of order η $(0\leq \eta < p)$ *in* $|z| \leq r_2$where

$$r_2 = inf_{k\geq n+p}\left(\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)^{\frac{1}{k-p}} \ldots\ldots\ldots(4.5)$$

The result is sharp the extremal function given by (2.4)

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \eta \quad \text{for} \quad |z| \leq r_2 \;\ldots\ldots\ldots(4.6)$$

where r_2 given by (4.5). From definition (1.2) that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}a_k|z|^{k-p}}$$

Thus

$$\left|\frac{zf'(z)}{z^{p-1}} - p\right| \leq p - \eta$$

If

$$\sum_{k=n+p}^{\infty}\left(\frac{k-\eta}{p-\eta}\right) a_k |z|^{k-p} \leq 1\ldots\ldots\ldots(4.7)$$

By using theorem 1 , (4.7) will be true if

$$\left(\frac{k-\eta}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)$$

$$|z| \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right)^{\frac{1}{k-p}} \quad k \geq n+p, n \in N \;\ldots\ldots (4.8)$$

Corollary 4. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*\ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$Then $f(z)$ is in p-valent convex of order η (0 ≤ η < p) in $|z| \leq r_3$, where

$$r_3 = inf_{k\geq n+p}\left\{\frac{[A+B(l-\beta)]p(p-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right\}^{\frac{1}{k-p}}$$

Sharpness is hold, with the extremal given by (2.4).

5.Closure theorems

Theorem 5. Let $\mu_j \geq 0\ for\ j = 1,2,\ldots.,m\ and\ \sum_{j=1}^{m}\mu_j \leq 1$, if function Body Math $f_j(z)$ defined by

$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k\ (a_{k,j} \geq 0, j = 1,2,\ldots,m)$ …(5.1)

are in the class $T_n^*\ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ for $j = 1,2,\ldots,m$ then the function f(z) defined by

$$f(z) = z^p - \sum_{k=p+n}^{\infty}\left(\sum_{j=1}^{\infty}(\mu_j a_{k,j}) z^k\right) \;\ldots\ldots (5.2)$$

Is also in the class $T_n^*\ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$

Proof :

Since $f_j(z)$ is in the class $T_n^*\ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then by theorem 1 that

$\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)a_{k,j} \leq \lambda(l-\beta)(1-\alpha)\delta(p-2,q)$ …….. (5.3)

For every j=1,2,… m Hence

$$\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)\left(\sum_{j=1}^{\infty}\mu_j a_{k,j}\right)$$

$$= \sum_{j=1}^{\infty} M_j \left(\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)a_{k,j}\right)$$

$$\leq \sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)a_{k,j}\sum_{j=1}^{\infty}\mu_j$$

$$= \lambda(l-\beta)(1-\alpha)\delta(p-2,q)$$

From theorem 1 , it follows that

$f(z)\epsilon\, T_n^*\ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ and so this completes the proof of theorem 5.

Corollary 5. The class $T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$ is closed under convq linear combination

Proof :

Let the function $f_j(z)(j=1,2)$ be given by (5.1) be in the class $T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$. It is sufficient to show that the function $f(z)$ defined by

$$f(z)=\mu f_1(z)+(1-\mu)f_2(z)$$

is in the class $T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$ .But , taking m=2 , $c_1=\mu$, $c_2=1-\mu$ in theorem 5 , We have the corollary

Theorem 6. Let $f_{p+n-1}(z)=z^p$ and

$$f_k(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k\ ,k\geq p+n \quad \ldots\ldots.(6.1)$$

Then $f(z)$ is in the class $T_n^*\ (p,q;A,B,\lambda,\alpha,l,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=p+n-1}^{\infty}M_kf_k(z)\ \ldots\ldots\ldots.(6.2)$$

Where $\mu_k\geq 0\ and\ \sum_{k=p+n-1}^{\infty}\mu_k=1$

Proof :

Assume that

$$f(z)=\sum_{k=p+n-1}^{\infty}\mu_kf_k(z)$$

$$=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\mu_kz^k\ \ldots\ldots(6.3)$$

Then it follows that

$$\sum_{k=p+n}^{\infty}\left(\frac{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)*\left(\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\mu_kz^k\right)$$

$$\leq\sum_{k=p+n}^{\infty}\mu_k=(1-\mu_{p+n-1})\leq 1$$

Hence by theorem 1, we have

$f(z)\in T_n^*\ (p,q;A,B,\lambda,\alpha,l,\beta)$.

Conversely , assume that the function $f(z)$defined by (1.2) belongs to the class

$T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$, then

$$a_k\leq\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k$$

Setting

$$\mu_k=\frac{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_k$$

Where

$$\mu_{p+n-1}=1-\sum_{k=p+n}^{\infty}\mu_k$$

We can see that $f(z)$ can be expressed in the form (5.5) ,this completes the proof of theorem 6

Corollary 6. The extreme point of the class $T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$ are the function $f_p(z)=z^p$ and

$$f_k(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k\quad k\geq p+n$$

6.Modified Hadamard products

Assume function $f_j(z)(j=1,2)$ defined by (5.1) the Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z^p-\sum_{k=p+n}^{\infty}a_{k,1}a_{k,2}z^k=(f_1*f_2)(z)\ \ldots.(6.1)$$

Theorem 7. Let the function $f_j(z)(j=1,2)$ defined by (5.1) be in the class $T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$, then $(f_1*f_2)(z)$ be in the class $T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)$,where

$$\sigma=1-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)](p+1)(p+n-1)\delta(p+n-2,q)}\quad (n\in N)\ \ldots..(6.2)$$

The result is sharp for the function $f_j(z)$(j = 1, 2) defined by

$$f_j(z)=z^p+\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)](p+1)(p+n-1)\delta(p+n-2,q)}z^{p+q}(6.3)$$

Proof : Depending the technique used earlier by Schild and Silverman [7], we must to show the largest σ such that

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1))\delta(k-1,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,1}a_{k,2}\leq 1\ \ldots\ldots(6.4)$$

We have $f_j(z)\in T_n^*\ (p,q,A,B,\lambda,\alpha,l,\beta)\ (j=1,2)$ then

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,1}\leq 1\ \ldots\ldots(6.5)$$

and

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,2}\leq 1\ \ldots\ldots(6.6)$$

By using Cauchy Scharz inequality , we have

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\sqrt{a_{k,1}a_{k,2}}\leq 1\ \ldots\ldots(6.7)$$

It is sufficient to show that

$$\frac{1}{1-\sigma}a_{k,1}a_{k,2}\leq\frac{1}{(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}\quad\ldots\ldots..(6.8)$$

or

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{1-\sigma}{1-\alpha}\ldots\ldots..(6.9)$$

Hence in night of the inequality (6.9), it is sufficient to prove that

$$\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\leq\frac{(1-\sigma)}{(1-\alpha)}\quad(k\geq p+n)\ldots\ldots..(6.10)$$

From (6.10) we have

$$\sigma\leq 1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ldots\ldots..(6.11)$$

In the next , we defined the function $R(k)$by

$$R(k)=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ldots\ldots.(6.12)$$

We note that $R(k)$ is an in creasing function of k (k$\geq p+n$), therefore , we caclud that

$$\sigma\leq R(p+n)=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}\ \ldots\ldots(6.13)$$

The proof is completes

Putting β = 1 in Theorem 7, we obtain the following corollary.

Corollary 7. Let the functions $f_j(z)(j = 1,2)$defined by (1.2) be in the class T_n^* (p, q; A , B , λ , α , l) Then where $\gamma = 1$

The result is sharp.

Corollary 8. For $f_1(z) and\ f_2(z)$ as in Theorem 7 , the function

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}a_{k,2}}\ z^k$$

belongs to the class $T_n^*\ (p, q; A\ , B\ , \lambda\ , \alpha\ , l)$.

This result follows from the Cauchy-Schwarz inequality (6.7). It is sharp for the same functions as in Theorem 7.

Theorem 8. Let the functions $f_j(z)(j = 1,2)$ defined by (5.1) be in the class$T_n^*\ (p, q; A\ , B\ , \lambda\ , \alpha\ , l)$. Then the function

$$h(z) = z^p - \sum_{k=p+n}^{\infty} \sqrt{\left(a^2{}_{k,1}a^2{}_{k,2}\right)}\ z^k \ \ldots\ldots.(6.14)$$

belongs to the class $T_n^*\ (p, q; A\ , B\ , \lambda\ , \varsigma, l, \beta), where$

The result is sharp for the functions $f_j(z)(j = 1,2)$ defined by (6.3)

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=p+n}^{\infty} \left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2 a^2{}_{k,1} \leq$$

$$\sum_{k=p+n}^{\infty} \left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} a^2{}_{k,1}\right]^2 \leq 1 \ldots.(6.16)$$

and

$$\sum_{k=p+n}^{\infty} \left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2 a^2{}_{k,2} \leq$$

$$\sum_{k=p+n}^{\infty} \left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} a^2{}_{k,2}\right]^2 \leq 1 \ldots.(6.17)$$

It follows from (6.16) and (6.17) that

$$\sum_{k=p+n}^{\infty} \frac{1}{2}\left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2 \left(a^2{}_{k,1}a^2{}_{k,2}\right) \leq$$

1Therefore, we need to find the largest ζ such that

$$\frac{\lambda(l-\beta)\delta(p-2,q)(1-\zeta)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \leq$$

$$\frac{1}{2}\left[\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2 \ldots\ldots\ldots.(6.19)$$

that is, that

$$\zeta \geq 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \ldots\ldots\ldots.(6.20)$$

since

$$D(k) = 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \ldots\ldots(6.21)$$

is an increasing function of $k(k \geq p+n)$, we readily have

$$\zeta \geq D(p+n)$$

$$= 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)} \ldots(6.22)$$

and Theorem 8 follows at once.

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