ON ESSENTIAL (T-SMALL) SUBMODULES

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***Abstract*:** Let M be an R-module and T be a submodule of M. A submodule K of M is called ET-small submodule of M(denoted by $K\ll_{ET}M$) , if for any essential submodule X of M such that $T\subseteq K+X$ implies that $T\subseteq X$. We study this mentioned definition and we give many properties related with this type of submodules.

***Keywords*:**T-small submodule, T-maximal submodule, T-Radical submodule, ET-small submodule, ET-maximal submodule, ET- Radical submodule.

1. Introduction

Throughout this paper R is a commutative ring with identity and M a unitary R-module. A proper submodule N of M is called small ($N \ll M$), if for any submodule K of M ($K \leq M$) such that $K + N = M$ implies that $N = M$. A submodule N of M is essential ($K \leq_e M$) if $K\cap L= 0$, then $L= 0$, for every $L\leq M$ [1] . A submodule N of M is closed ($N \leq_c M$) if N has no proper essential extensions inside M that is, if the only solution of the relation $N \leq_e K \leq_e M$ is N=K[2]. The submodule N of M is called an essential- small ($N \ll_e M$)submodule of M , if for every essential submodule T of M such that $M = N + T$ implies $T = M$ [3].

In [4] the authors introduced the concept of small submodule with respect to an arbitrary submodul ,that a submodule K of M is called T-small in M , denoted by $K \ll T\ M$, in case for any submodule X of M, such that $T\subseteq K+X$, implies that $T\subseteq X$.

In this work we introduce essential T-small (ET-small) submodule, where an R-module M and T be a submodule of M. A submodule K of M is called ET-small submodule of M (denoted by $K\ll_{ET} M$), if for any essential submodule X of M such that $T\subseteq K+X$ implies that $T\subseteq X$. In the first section , we give the fundamental properties of of ET-small submodules, Also we give many relations between ET-small submodule and other kinds of small submodules.

In the second section, we introduce essential T-maximal(ET-maximal) submodules and the essential T-radical (ET-radical) submodules of M denoted by $Rad_{ET}M$, We give the fundamental properties of this concepts.

2. Essential T-small submodule.

Definition2.1:Let M be an R-module and let T be a submodule of M .A submodule K of M is called ET-small submodule of M (denoted by $K\ll_{ET} M$), if for any essential submodule X of M such that $T\subseteq K+X$ implies that $T\subseteq X$.

Remarks and Examples 2.2:

1. Consider Z_6 as Z-module .Let $T=\{\bar{0},\bar{3}\}$, $K=\{\bar{0},\bar{2},\bar{4}\}$.The only essential submodule of Z_6 is Z_6 if $T\subseteq K+Z_6$,then $T\subseteq Z_6$ .Thus$K\ll_{ET}Z_6$.

2.It is clear that Every T-small submodule of M is ET-small submodule of M but the converse is not true as for the following Consider Z_{24} as Z-module and Let $T=\{\bar{0},\bar{8},\overline{16}\}$, $N=8Z_{24}$ the only essential submodule in Z_{24} are $2Z_{24}$, $4Z_{24}$ and Z_{24} , $T=8Z_{24}\subseteq 8Z_{24}+2Z_{24}$ and $8Z_{24}\subseteq 2Z_{24}$,also $8Z_{24}\subseteq 8Z_{24}+4Z_{24}$, $8Z_{24}\subseteq 4Z_{24}$ and $8Z_{24}\subseteq Z_{24}$ Then $8Z_{24}$ ET-small submodule of Z_{24} which is not T-small submodule of Z_{24} since $8Z_{24}\subseteq 8Z_{24}+3Z_{24}but8Z_{24}\not\subseteq 3Z_{24}$.

3. Let M be an R- module and T=0.Then every essential submodule of M is ET-small in M.

4. Let M be an R-module and T=M .Then $N\ll_{ET}M$ if and only if $N\ll_e M$.

Proposition 2.3: Let M be an R-module and let T,H and L be submodules of M such that $T \leq N$ and $H \leq N \leq M$ and $N\ll_e M$. If $H\ll_{ET} M$, then $H\ll_{ET} N$.

Proof: Let H be ET-submodules of M and X be an essential submodule of N such that $T\subseteq H+X$ .since $X \leq_e N$ and $N\leq_e M$ so $X \leq_e M$[2], then $H\ll_{ET} M$, and $T\subseteq X$.

Proposition 2.4: Let M be an R-module with submodules $N \leq H \leq M$ such that $T\leq H$ .If $N\ll_{ET} H$, then $N\ll_{ET} M$.

Proof*:* Suppose that $T\subseteq N+X$, for any essential submodule X of M .Since $T\subseteq H$, then $T=T\cap H\subseteq(N+X)\cap H=N+(X\cap H)$ by modular law , since $X \leq_e M$ and $H \leq_e H$, then $(X\cap H) \leq_e (M\cap H) =H$ [2], and $N\ll_{ET} H$, then $T\subseteq X$. Thus $N\ll_{ET} M$.

Proposition 2.5: Let M be an R-module and Let T, N_1 and N_2 be a submodules of M, Then $N_1\ll_{ET} M$ and $N_2\ll_{ET} M$ if and only if $N_1+N_2\ll_{ET} M$.

Proof*:* Suppose that $N_1\ll_{ET} M$ and $N_2\ll_{ET} M$ and Let $T\subseteq (N_1+N_2)+X$, for any essential submodule X of M, then $T\subseteq N_1+(N_2+X)$, since $X \leq_e M$ and $N_2\leq_e N_2$, then $N_2+X \leq_e M$ [2] and $N_1\ll_{ET} M$ Then $T\subseteq N_2+X$, since $N_2\ll_{ET} M$ Then $T\subseteq X$. Conversely, let $N_1+N_2\ll_{ET} M$, to show that $N_1\ll_{ET} M$ and $N_2\ll_{ET} M$, Suppose that $T\subseteq N_1+X$, for any essential submodule X of M , since $N_1\subseteq N_1+N_2$,so $T\subseteq N_1+N_2+X$, but $N_1+N_2\ll_{ET} M$, so $T\subseteq X$, thus $N_1\ll_{ET} M$, and the same we have $N_2\ll_{ET} M$.

Proposition 2.6: Let M be an R-module and Let H be a submodule of M .If $\{T_i\}_{i\in I}$ be a family set of submodules of M such that $H \ll_{ET_i} M$, for each $i \in I$, then $H \ll_{E(\sum_{i\in I}T_i)} M$.

Proof:Let ($\sum_{i\in I} T_i$)$\subseteq$ H+X, for any essential submodule X of M. then for each $i \in I$, $T_i \subseteq H+X$ and by hypothesis $T_i \subseteq X$, thus ($\sum_{i\in I} T_i$)$\subseteq$ X.

Proposition2.7: Let M and N be any R-modules and f : $M \to N$ be a homomorphism .If T and H are submodules of M such that $H\ll_{ET} M$,then $f(H)\ll_{Ef(T)} N$.

Proof: Let $f(T) \neq 0$ and $f(T)\subseteq f(H)+X$, for any essential submodule X of N .to show $T\subseteq H+ f^{-1}(X)$. let $t \in T$, then $t=h+w$,for some $h\in H$ and $b\in f^{-1}(X)$ .Hence $f(t)=f(h + b)= f(h)+f(b)$ .Thus $f(t)-f(h) = f(b)$, thus $f(t-h) = f(b) \in X$ and so$(t-h) \in f^{-1}(X)$ implies that $t \in H+ f^{-}$

$^{1}(X)$.since $X \leq_e N$ Thus $f^{-1}(X) \leq_e M$ [2] and $H \ll_{ET} M$,therefore $T \subseteq f^{-1}(X)$ .Thus $f(T) \subseteq X$.

Proposition 2.8: Let M be an R-module and Let T, H and N be submodules of M such that $H \leq N \leq M$ and $H \leq T$. if $N \ll_{ET} M$ then $H \ll_{ET} M$ and $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$.

Proof: Let $N \ll E_{GT} M$ .To show that $H \ll_{ET} M$, let $T \subseteq H+X$, for any essential submodule X of M .but $H \leq N \leq M$, so $T \subseteq N+X$ .then $T \subseteq X$ .Thus $H \ll_{ET} M$. Now to show that $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$, let $\frac{T}{H} \subseteq \frac{N}{H} + \frac{X}{H}$, for any essential submodule $\frac{X}{H}$ of $\frac{M}{H}$ such that $H \subseteq X$. Then $\frac{T}{H} \subseteq \frac{N+X}{H}$ so $T \subseteq N+X$, Since $\frac{X}{H} \leq_e \frac{M}{H}$ then $X \leq_e M$ [3] , and $N \ll_{ET} M$, then $T \subseteq X$ Thus $\frac{T}{H} \subseteq \frac{X}{H}$.

Proposition2.9: Let M be an R-module and Let T, H and N be submodules of M such that $H \leq N \leq M$ and $H \leq T$ and $H \leq_c M$, if $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$ then $N \ll_{ET} M$.

Proof: Let $H \ll_{ET} M$ and $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$,to show that $N \ll_{ET} M$, let $T \subseteq N+X$, for any essential submodule X of M and $H \subseteq X$. Now $\frac{T}{H} \subseteq \frac{N+X}{H} = \frac{N}{H} + \frac{X}{H}$, since $X \leq_e M$ and $H \leq_c M$ then $\frac{X}{H} \leq_e \frac{M}{H}$ [3]. But $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$, so $\frac{T}{H} \subseteq \frac{X}{H}$ implies that $T \subseteq X$ .thus $N \ll_{ET} M$.

Corollary2.10: Let M be an R-module and Let K and H be submodules of M such that $K \ll_{EH} M$ and $H \ll_{EK} M$. Then $(H \cap K) \ll_{E(K+H)} M$.

Proof: Let $K \ll_{EH} M$ and $H \ll_{EK} M$, since $(H \cap K) \leq H$ and $(H \cap K) \leq K$, by Proposition (1.4),$(H \cap K) \ll_{EK} M$ and $(H \cap K) \ll_{EH} M$. Also by Proposition (1.6) we get $(H \cap K) \ll_{E(K+H)} M$.

Proposition 2.11: Let M be an R-module and Let T , K ,H and B be submodules of M such that $K \leq H \leq B \leq M$, $K \leq_c M$ and $H \leq_c M$. Then $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ if and only if $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$ and $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$.

Proof: Let $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$.To show that $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$, let $\frac{T+H}{H} \subseteq \frac{B}{H} + \frac{X}{H} = \frac{B+X}{H}$,for any essential submodule $\frac{X}{H}$ of $\frac{M}{H}$ and $H \subseteq X$. Hence $T \subseteq T+H \subseteq B+X$ .So $\frac{T+K}{K} \subseteq \frac{B+X}{K}$ .Therefore $\frac{T+K}{K} \subseteq \frac{B}{K} + \frac{X}{K}$ .since $X \leq_e M$ and $K \leq_c M$ then $\frac{X}{k} \leq_e \frac{M}{k}$ [3] But $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$,therefore $\frac{T+K}{K} \subseteq \frac{X}{K}$ .So $T \subseteq T+K \subseteq X$ and hence $\frac{T+H}{H} \subseteq \frac{X}{H}$ .Thus $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$. To show that $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .Let $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K}$,for any essential submodule $\frac{X}{K}$ of $\frac{M}{K}$, $K \subseteq X$. Then $\frac{T+K}{K} \subseteq \frac{H+X}{K}$ and hence $T \subseteq T+K \subseteq H+X \subseteq B+X$ implies that $\frac{T+K}{K} \subseteq \frac{B+X}{K} = \frac{B}{K} + \frac{X}{K}$, Since $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$, then $\frac{T+K}{K} \subseteq \frac{X}{K}$ .Thus $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .Conversely, let $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$ *and* $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .To shows that $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .Let $\frac{T+K}{K} \subseteq \frac{B}{K} + \frac{X}{K}$, for any essential submodule $\frac{X}{K}$ of $\frac{M}{K}$.Then $\frac{T+K}{K} \subseteq \frac{B+X}{K}$ and hence $T \subseteq T+K \subseteq B+X$ .so $T+H \subseteq B+X+H$ implies that $\frac{T+H}{H} \subseteq \frac{B+X+H}{H} = \frac{B+X}{H}$. therefore $\frac{T+H}{H} \subseteq \frac{B}{H} + \frac{X}{H}$, since $X \leq_e M$ and $H \leq_c M$ then $\frac{X}{H} \leq_e \frac{M}{H}$ [3] and $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$, therefore $\frac{T+H}{H} \subseteq \frac{X}{H}$. Then $T+H \subseteq X$ But $T+k \subseteq T+H$ and $X \subseteq X+H$, so $T+k \subseteq T+H \subseteq X \subseteq X+H$ therefore $T+k \subseteq X+H$, so $\frac{T+K}{K} \subseteq \frac{X+H}{K} = \frac{X}{K} + \frac{H}{K}$, since $X \leq_e M$ and $K \leq_c M$ then $\frac{X}{K} \leq_e \frac{M}{K}$ and $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ then $\frac{T+K}{K} \subseteq \frac{X}{K}$.Thus $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$.

Proposition2.12: Let $M=M_1 \oplus M_2$ be an R-module such that $R=Ann(M_1)+Ann(M_2)$ .If $N_1 \ll_{ET1} M_1$ and $N_2 \ll_{ET2} M_2$, then $N_1 \oplus N_2 \ll_{E(T1 \oplus T2)} M$.

Proof: Let $T_1 \oplus T_2 \subseteq N_1 \oplus N_2 + X$, for any essential submodule X of M .Since $R=Ann(M_1)+Ann(M_2)$ .Then, by the same argument of the prove of [6, prop. 4.2, ch 1] $X= X_1 \oplus X_2$,for any essential submodule X_1 of M_1 and submodule X_2 of M_2 .Hence $T_1 \oplus T_2 \subseteq N_1 \oplus N_2 + X_1 \oplus X_2$,implies that $T_1 \oplus T_2 \subseteq (N_1+X_1) \oplus (N_2+X_2)$ .to show that $T_1 \subseteq N_1+X_1$ and $T_2 \subseteq N_2+X_2$ .let $t_1 \in T_1$ and $t_2 \in T_2$ then $t_1+t_2 \in T_1 \oplus T_2 \subseteq (N_1+X_1) \oplus (N_2+X_2)$, so $t_1 \in (N_1+X_1)$ and $t_1 \in (N_2+X_2)$, then $T_1 \subseteq N_1+X_1$ and $T_2 \subseteq N_2+X_2$, Since $N_1 \ll_{ET1} M_1$ and $N_2 \ll_{ET2} M_2$, then $T_1 \subseteq X_1$ and $T_2 \subseteq X_2$ and hence $T_1 \oplus T_2 \subseteq X_1 \oplus X_2 = X$ .Thus $N_1 \oplus N_2 \ll_{E(T1 \oplus T2)} M$.

Recall that an R-module M is called a fully stable module if for each submodule K of M and for each R-homomorphism f from M into K, $f(K) \subseteq K$ [5].

Proposition2.13: Let $M=\oplus_{i \in I} M_i$ be a fully stable module .If $K_i \ll_{E(Ti)} M_i$, for each $i \in I$, then $\oplus_{i \in I} K_i \ll_{E(\oplus_{i \in I} T_i)} \oplus_{i \in I} M_i$.

Proof: Let $M=\oplus_{i \in I} M_i$ be a fully stable module and $K_i \ll_{E(Ti)} M_i$, for each $i \in I$ .To show that $\oplus_{i \in I} K_i \ll_{E(\oplus_{i \in I} T_i)} \oplus_{i \in I} M_i$ .Let $(\oplus_{i \in I} T_i) \subseteq (\oplus_{i \in I} K_i)+X$, for any essential submodule X of M .Claim that $X=\oplus_{i \in I}(X \cap M_i)$ .To show that, for each $i \in I$ let $P_i : M \to M_i$ be The projection map and let $x \in X$,then $x \in \oplus_{i \in I} M_i$ and hence $x=\sum_{i \in I} x_i$ where $x_i \in M_i$, $\forall\, i \in I$ and $x_i \neq 0$ for at most a finite number of $i \in I$ .Since M is fully stable, then $P_i(x) \in X$, $\forall\, i \in I$ .Now $P_i(x) = P_i(\sum_{i \in I} x_i) = x_i \in (X \cap M_i)$ and hence $x= (\sum_{i \in I} x_i) \in \oplus_{i \in I}(X \cap M_i)$ .Thus $X \subseteq \oplus_{i \in I}(X \cap M_i)$ .Clearly $\oplus_{i \in I}(X \cap M_i) \subseteq X$ .Thus $K= \oplus_{i \in I}(K \cap M_i)$ .Now $\oplus_{i \in I} T \subseteq (\oplus_{i \in I} K_i)+(\oplus_{i \in I}(X \cap M_i)) = \oplus_{i \in I}(K_i+(X \cap M_i))$ .Therefore $T_i \subseteq K_i+(X \cap M_i)$,for each $i \in I$ .Since $K_i \ll_{E(Ti)} M_i$,then $T_i \subseteq (X \cap M_i)$ and hence $\oplus_{i \in I} T \subseteq \oplus_{i \in I}(X \cap M_i)=X$. thus $\oplus_{i \in I} K_i \ll_{E(\oplus_{i \in I} T_i)} \oplus_{i \in I} M_i$.

Recall that the annihilator of M $Ann(M) =\{r \in R \mid rM = 0\}$ [6], M is a faithful module if $Ann(M) = 0$. M is a multiplication module if for each submodule N of M, there exists an ideal I of R such that N=IM [7].

Proposition2.14: Let M be a finitely generated, faithful and multiplication module and let I, J be ideals in R .Then $I \ll_{EJ} R$ if and only if $IM \ll_{E(JM)} M$.

Proof: Let $I \ll_{EJ} R$ .To show that $IM \ll_{E(JM)} M$ .Let $JM \subseteq IM+X$, for any essential submodule X of M .Since M is multiplication module, then X=KM, for some ideal K of R and hence $JM \subseteq IM+KM= (I+K)M$ .since M be a finitely generated, faithful and multiplication module, therefore M is a cancellation module, by [9] .then $J \subseteq I+K$ since $K \leq_e R$ .Since $I \ll_{EJ} R$,then $J \subseteq K$ .Hence $JM \subseteq KM=X$ .Thus $IM \ll_{E(JM)} M$. Conversely, let $IM \ll_{E(JM)} M$ .To show that $I \ll_{EJ} R$ .Let K be essential ideal of R such that $J \subseteq I+K$ .Since M is multiplication module, then

JM⊆IM+KM .But IM$\ll_{E(JM)}$M, therefore JM⊆KM .So J⊆K .Thus I$\ll_{EJ}$ R.

3. Essential T-Radical of M.Recall that if M an R-module and T be a submodule of M. A submodule K of M is called T-maximal submodule of M if is $\frac{T+K}{K}$ simple [4]. In this section, we introduce the definitions of ET-maximal submodules and ET-radical of a module as a generalization of T-maximal submodules and T-radical of a module and we discuss some of the basic properties of this concepts.

Definition 3.1: Let M be an R-module and let T be a submodule of M .An essential submodule K of M is called essential T-maximal(ET-maximal) submodule of M if $\frac{T+K}{K}$ is simple.

Remarks and examples 3.2:

1.If M is a unform R-module M and let K be a submodule of a module M, then K is ET-maximal submodule of M if and only if K is T- maximal submodule of M.

2.If T a submodule of M then every ET-maximal submodule of M is T- maximal submodule of M but the converse is not true as the following example.

Consider Z_6 as Z-module .Let T=$\{\bar{0},\bar{2},\bar{4}\}$ and K=$\{\bar{0},\bar{3}\}$.Then K is T-maximal submodule of Z_6 , where $\frac{\{\bar{0},\bar{2},\bar{4}\}+\{\bar{0},\bar{3}\}}{\{\bar{0},\bar{3}\}}=\frac{Z_6}{\{\bar{0},\bar{3}\}}\cong\{\bar{0},\bar{2},\bar{4}\}$ is simple, but K is not ET-maximal submodule of Z_6,since K is not essential submodule of Z_6.

Since every ET-maximal submodule of M is T-maximal submodule .The following we get without prove since the prove is as the same way on [4],[8]

Proposition 3.3: Let M be an R-module and($0 \neq$T) be a proper finitely generated submodule of M and let

A = {L ≤ M | L $\ll_{ET}$M and L+K ⊆ T +K, for all ET-maximal submodule K of M} and

B = {K ≤ M | K is an ET-maximal submodule of M}. Then $\sum_{L\in A} L=\bigcap_{K\in B} K$.

Proposition 3.4: Let M be an R-module and be a finitely generated submodule of M and a∈M Then Ra is not ET-small submodule of M if and only if there exists H is ET-maximal submodule of M such that a∉H and T ⊆ Ra + H.

Proposition 3.5: Let M and N be an R-modules and f : M → N be an R-homomorphism. If T is a submodule of M and K is an ET-maximal submodule of M such that kerf ⊆ K, then f(K) also is an Ef(T)-maximal submodule of N.

Proposition 3.6: Let M and N be an R-modules and f : M → N be an R- epimorphism . If T is a submodule of M and K is an Ef(T)-maximal submodule of N, then f^{-1}(K) also is an ET-maximal submodule of M.

Proposition 3.7: Let H and T be submodules of a module M such that T is finitely generated and T⊈H .Then there exists a ET-maximal submodule of M containing H.

Definition 3.8: Let M be an R – module the intersection of all essential T-maximal submodules of M is called a essential T-Radical of M (denoted by Rad_{ET}(M)) .If M has no ET-maximal submodule , then Rad_{ET}(M) = T .

Remarks and Examples 3.9:

1. If M be an uniform R-module then $Rad_{ET}(M) = Rad_T(M)$.

2.If.T=M.then.$Rad_{ET}(M)=Rad_e(M)$.

3.Consider Z_6 as Z-module .Let T=Z_6 and K_1 = Z_6 are ET-maximal submodules of Z_6 , therefore Rad_{GT}(Z_6) = Z_6 .

4. Consider Z_4 as Z-module .Let T= Z_4 and K= $\{\bar{0},\bar{2}\}$, then K is the only ET-maximal submodule of Z_4 .To show that, $\frac{Z_4+\{\bar{0},\bar{2}\}}{\{\bar{0},\bar{2}\}}\cong\{\bar{0},\bar{2}\}$ is a simple .Thus Rad_{ET} Z_4= Rad_T Z_4=$\{\bar{0},\bar{2}\}$.

5.Consider Z_{p^∞} as Z-module .Let T= Z_{p^∞}, then Z_{p^∞} has no ET-maximal submodule and hence $Rad_{ET}Z_{p^\infty}$= Z_{p^∞}.

Proposition 3.10: Let M be an R-module and let T be a finitely generated submodule of a module M .Then $Rad_{ET}(M) \ll_{ET}$ M.

Proof: Assume that T ⊆ Rad_{ET}M +X, for any essential submodule X of M .to show that T ⊆ X suppose that T ⊈ X .Then by *Proposition* (2.7) , there exists a ET-maximal submodule K of M such that X ⊆ K .Therefore T ⊆ Rad_{ET}M + X ⊆ K .implies that T ⊆ K, so $\frac{T+K}{K}$ = 0 which contradicts the T-maximality of K. Thus T ⊆ X, Thus $Rad_{ET}(M) \ll_{ET}$ M.

Lemma 3.11: Let M be an R- module and let T be a finitely generated submodule of a module M and m ∈ M such that R_m+H ⊆T+H, for all ET-maximal submodule H of M, then $R_m \ll_{ET}$M iff m ∈Rad_{ET} (M) .

Proof: Let $R_m \ll_{ET}$M and R_m+ H ⊆ T+H, for all ET-maximal submodule H of M , By Proposition (2.3) then R_m ∈ A,where A = {L ≤ M | L$\ll_{ET}$M and L+H⊆T+H, for all ET-maximal submodule H of M} . Hence R_m ⊆ Rad_{ET}M. For the converse, let $m\in Rad_{ET}$M . To show that $R_m \ll_{ET} M$. Suppose that R_m is not ET-small submodule M . By Proposition (2.4), then there exists H is a ET-maximal submodule of M with $m\notin$ H then $m\notin Rad_{ET}$M which is a contradiction .Thus R_m is a ET-small submodue of M.

Proposition 3.12: Let M and N be an R-modules and f : M → N be an R-epimorphism such that Kerf ⊆ Rad_{ET}M. Then f(Rad_{ET}M) = Rad $_{Ef(T)}$N.

Proof: Since f is epimorphism, by *Proposition* (2.5) and Proposition(2.6),we,have f(Rad_{ET}M) = f($\bigcap_{K\in A} K$) = $\bigcap_{f(k)\in B} f(K)$ = $Rad_{Ef(T)}$N, where, A = {K ≤ M | K is an ET-maximal submodule of M} and B={f(K) ≤ N| f(K) is an Ef(T)-maximal submodule of N}.

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