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PROCEEDINGS

Numerical reconstruction of time-dependent thermal conductivity for heat equation from non-local overdetermination conditions

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Abstract: Recovery of time-dependent thermal conductivity has been numerically investigated. The problem of identification in one-dimensional heat equation from Cauchy boundary data and mass/energy specification has been considered. The inverse problem recasted as a nonlinear optimization problem. The regularized least-squares functional is minimised through *lsqnonlin* routine from MATLAB to retrieve the unknown coefficient. We investigate the stability and accuracy for numerical solution for two examples with various noise level and regularization parameter.

Keywords: Inverse problem; Finite difference method; nonlinear optimization, heat equation, regularization, coefficient identification problem.

1. Introduction

The concept of so-called inverse/backward problems have been dominated the research in late twentieth century due to wide range of applications; such as in engineering, geo- physics, economics and ecology [1].

During the consideration of inverse problems, the choice of overdetermination/additional conditions play an important role in the proofing the existence and uniqueness of the solution for example see, [2, 3, 4].

We investigate the numerical reconstruction of time-dependent thermal conductivity with respect to initial and non-homogeneous Dirichlet boundary conditions. Whilst, the overdetermination conditions are the energy/mass specification and the heat flux difference.

The organization of the paper as follows. In next section, the mathematical formulation and the unique solvability conditions are stated. In Section 3, the Crank-Nicolson FDM scheme for the direct problem has been presented and developed [5]. While, in Section 4 we consider the numerical solutions for inverse problems based on finding the quasi-solution for associated nonlinear optimization problem. Section 5, devoted to the numerical results and discussion. Finally, the conclusions are highlighted in Section 6.

2. Mathematical formulation

Consider the fixed parameters h > 0 and T > 0 which represent the length of a finite slab and time, respectively. The solution domain denoted by

$$D=\{(x,t):\ 0<x<h;\ 0<t\leq T\}.$$

In this paper, we consider the heat equation of the form

$$u_t = a(t)u_{xx} \qquad (x,t) \in D \qquad (1)$$

where $a(t) > 0$ is coefficient and $u(x, t)$ is the temperature. The heat capacity is taken to be unity and therefore the coefficient a(t) represent the time-dependent thermal conductivity. Equation (1) has to be solved subject to initial condition

$$u(x,0) = \varphi(x) \qquad 0 \leq x \leq h \qquad (2)$$

and nonhomogeneous Dirichlet boundary conditions

$$u(0,t) = \mu_1(t) \quad u(h,t) = \mu_2(t) \quad 0 \leq t \leq T \qquad (3)$$

If the coefficient a is given the equations (1)–(3) formulate a direct Dirichlet problem for finding the unknown temperature $u(x,t)$. The outputs of interest which should be computed are energy/mass specification

$$\int_0^h u(x,t)dx = \mu_3(t) \qquad 0 \leq t \leq T \qquad (4)$$

and the difference of heat flux at the ends

$$u_x(h,t) - u_x(0,t) = \mu_4(t) \quad 0 \leq t \leq T \qquad (5)$$

If conductivity coefficient a is unknown, in this case, an inverse problem for coefficient identification should be solved.

The problem for identifying the coefficient a(t) has been considered in [6] with periodic boundary conditions and nonlocal overspecified data. In this paper, the consideration has been given to recovery the unknown coefficient under different boundary and overdetermination conditions as in equations (4) and (5). The unique solvability theorems for theses inverse problems are stated in the next subsections

2.1 Inverse problem 1 (IP1)

The IP1 requires determination of thermal conductivity a(t) > 0 together with the temperature u(x, t) satisfying the equations (1)–(4).

The unique solvability for IP1 was established in [7].

Theorem 1. *The inverse problem (1)–(4) is uniquely solvable if $\varphi(x) \in C^1[0, h]$, $\mu_i(t) \in C^1[0, T]$, $i = 1, 2$, $\varphi'(h - x) - \varphi'(x) \geq 0$ on $[0,h/2]$, $\mu_1'(t) + \mu_2'(t) \geq 0$ on $[0, T]$ and at least one of the functions $\mu_1'(t) + \mu_2'(t)$ and $\varphi'(h - x) - \varphi'(x)$ is not identically zero.*

Remark: notice that by differentiating equation (4) with respect to *t* and invoke (1) we have

$$\mu'_3(t) = \int_0^h u_t(x,t)dx = \int_0^h a(t)u_{xx}(x,t)dx \qquad (6)$$

after simple manipulation we obtain

$$a(t) = \frac{\mu'_3(t)}{u_x(h,t)-u_x(0,t)}, \qquad 0 \leq t \leq T, \qquad (7)$$

at $t = 0$

$$a(0) = \frac{\mu'_3(0)}{\varphi'(h)-\varphi'(0)}, \qquad (8)$$

provided that $\varphi'(h) - \varphi'(0) \neq 0$.

2.2 Inverse problem 2 (IP2)

The IP2 requires to solve the equations (1)–(3) and (5). Also, the unique solvability for IP2 was established in [7] and reads as follows.

Theorem 2. *Assume that the following conditions are satisfied*

1. $\mu_i \in C^1[0,T]$, i=1,2,4, $\varphi(x) \in C^2[0,h]$, $\varphi(0)+\varphi(h)=\mu_1(0)+\mu_2(0)$ and $\varphi'(h)-\varphi'(0) = \mu_4(0)$;
2. $\mu'_i(t)>0$, i=1,2, $\mu_4(t)\geq 0$ for $t\in[0,T]$, $\varphi''(x)>0$ on [0,h].

Then the inverse problem (1)–(3) and (5) possesses a solution.

Theorem 3. *Assume that the functions $\varphi(x)$ and $\mu_4(t)$ satisfy the conditions*

1. $\varphi(x) \in C^2[0,h]$, $\mu_4(t) \in C^1[0,T]$, $\varphi''(x) \geq 0$ on [0,h] and $\mu'_4(t)$ on [0,T];
2. at least one of the functions $\varphi''(x)$ and $\mu'_4(t)$ is not identically zero.

Then the solution of the inverse problem (1)–(3) and (5) is unique.

3. Direct problem

In this section, consider numerical solution for the direct initial boundary value problem given by equation (1)–(3). The Finite difference method (FDM) with Crank-Nicolson scheme [5], has been employed which is unconditionally stable and second-order accurate in space and time. In order to employ this scheme, denote $u(x_i,t_j) := u_{i,j}$ and $a(t_j) := a_j$, where $x_i = i\Delta x$ and $t_j = j\Delta t$, $\varphi(x_i)=\varphi_i$, $\mu_k(t_j)=\mu_{kj}$, for k=1,2,3,4, i=0,M, j=0,N, $\Delta x=h/M$ and $\Delta t=T/N$.

Considering the heat equation (1), the Crank-Nicolson method, discretise (1)–(3) as

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\Big(a_{j+1}\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} + a_j\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}\Big),$$
$$i = 1,(M-1), j = 0,(N-1) \qquad (9)$$

$$u_{i,0} = \varphi(x_i), \qquad i = 0, M \qquad (10)$$

$$u_{0,j} = \mu_1(t_j), u_{M,j} = \mu_2(t_j), \qquad j = 0, N. \qquad (11)$$

equations (9)–(11) can be discretise in the difference equation form

$$-A_{j+1}u_{i-1,j+1} + (1 + 2A_{j+1})u_{i,j+1} - A_{j+1}u_{i-1,j+1} = A_j u_{i-1,j} + (1 - 2A_j)u_{i,j} + A_j u_{i-1,j} \qquad (12)$$

for i=0,M−1, j =0,N−1 where $A_j = (\Delta t)a_j/(2(\Delta x)^2)$. At each time step t_{j+1} for j= 0,N−1 using Dirichlet boundary conditions the above difference equation can be expressed as (M − 1) × (M − 1) system of linear equations take the form

$$Cu_{j+1} = Ku_j + b$$

where $u_{j+1} = (u_{1,j+1}, u_{2,j+1}, ..., u_{M-1,j+1})^T$ for j = 0, N, $b = (b_1, b_2, ..., b_{M-1})^T$, C and K are tridiagonal matrices.

An example, consider the direct problem (1)–(3) with h = T = 1, and

$$\varphi(x) = u(x,0) = \exp(x) + \cosh(x) \qquad (13)$$

$$\mu_1(t) = u(0,t) = 2\exp(t^3 + t) \qquad (14)$$

$$\mu_2(t) = u(1,t) = (\exp(1) + \cosh(1))\exp(t^3 + t) \quad (15)$$

$$a(t) = 2t^2 + 1. \quad (16)$$

The true solution is given by

$$u(x,t) = (\exp(x) + \cosh(x))\exp(t^3 + t) \quad (17)$$

Figure 1 present the numerically obtained solution for *u(x,t)*. This figure indicates that an excellent agreement with true solution.

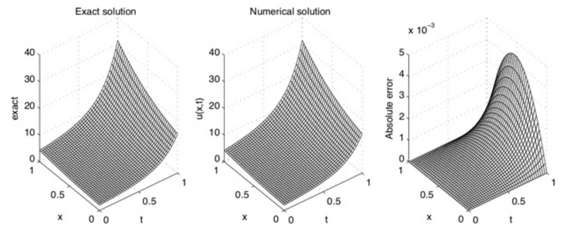

Figure 1*. the exact solution (left), numerical solution (middle), absolute error between them (right), for direct problem (1)-(3).*

On the other hand, outputs of interest equations (4) and (5), which are analytically given by

$$\mu_3(t) = (e + \sinh(1) - 1)\exp(t^3 + t), \quad t \in [0,1] \quad (18)$$

$$\mu_4(t) = (e + \sinh(1) - 1)\exp(t^3 + t), \quad t \in [0,1] \quad (19)$$

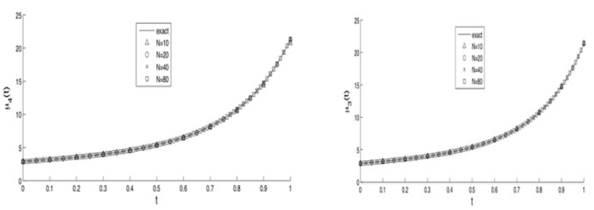

Figure 2. *The exact and numerical (left) $\mu_3(t)$ (right) $\mu_4(t)$ for M=N∈{10,20,40,80} for direct problem.*

Figure 2 explain that the true and numerical solutions for equations (4) and (5) are indistinguishable. The true solutions are given by equations (18) and (19), whilst the approximate results have been computed using the following second order finite-difference and trapezoidal rule formula.

$$\mu_3(t_j) = \int_0^1 u(x,t_j)dx = \frac{1}{2N}\left(\mu_1(t_j) + \mu_2(t_j) + 2\sum_{i=1}^{M-1} u_{i,j}\right), \quad j = 0, N \quad (20)$$

$$\mu_4(t_j) = \frac{4u_{M-1,j} - u_{M-2,j} - 3\mu_2(t_j)}{-2\,\Delta x} - \frac{4u_{1,j} - u_{2,j} - 3\mu_1(t_j)}{2\,\Delta x}, \quad j = 0, N, \quad (21)$$

To measure the accuracy of our numerical results, the root means square errors (*rmse*) between the numerical and the exact results for equations (4) and (5) are shown in the Table

M=N	10	20	40	80
$rmse(\mu_3)$	0.0243	0.0058	0.0014	3.2E-4
$rmse(\mu_4)$	0.2340	0.0562	0.0136	0.0033

1 which shows the mesh convergence,

$$rmse(\mu_k) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\mu_k^{approx} - \mu_k^{exact})^2} \quad k = 3,4 \quad (22)$$

Table 1 *.The (rmse) given by (22), between the true and numerical solution for (4) and (5) for M=N ∈ {10,20,40,80} for direct problem (1)-(3).*

In this table it can be seen that as M = N increase the *rmse* values decrease indicating the mesh convergence.

4. Numerical solutions of the inverse problems

Suppose the coefficient a(t) > 0 is unknown, in this case we deal with inverse problem. To solve this, one can look at the quasi-solution [8], given by solving the minimization of the least-squares gap, for nonlinear ill-posed and inverse problem. We employ the Tikhonov regularization method based on minimizing the following functional;

$$F(a) = \left\| \int_0^h u(x,t)dx - \mu_3(t) \right\|^2 + \lambda||a||^2, \quad (23)$$

for IP1 and the corresponding functional for IP2 given by

$$G(a) = \|u_x(h,t) - u_x(0,t) - \mu_4(t)\|^2 + \beta||a||^2, \quad (24)$$

where u solves (1)–(4) or (1)–(3) and (5), respectively, λ ≥ 0 and β ≥ 0 are regularization parameters and the norm is usually the L^2 [0, T]-norm. The discrete form of the functionals (23) and (24) are;

$$F(\boldsymbol{a}) = \sum_{j=1}^{N}\left[\int_0^h u(x,t_j)dx - \mu_3(t_j)\right]^2 + \lambda\sum_{j=1}^{N} a_j^2, \quad (25)$$

$$G(\boldsymbol{a}) = \sum_{j=0}^{N}\left[u_x(h,t_j) - u_x(0,t_j) - \mu_4(t_j)\right]^2 + \beta\sum_{j=0}^{N} a_j^2, \quad (26)$$

Clearly if λ = β = 0, the above equations yield the ordinary least-squares methods which is usually produce unstable solution for noisy input data. It is worth to mention that, the minimization of *F* or *G* subject to the physical constraints that the thermal conductivity is positive quantity. The minimization processes are accomplished using the MATLAB routine *lsqnonlin* from optimization toolbox, [9]. This routine based on Trust-Region- Reflection (TRR) to find the local minimization of sums of squares functions starting from initial guess.

The needed parameters for the routine are taken as follows;

- Solution Tolerance (xTol)=10^{-10},
- Function Tolerance (FunTol)=10^{-10},
- Initial guess ($a^0 = a(0)$) for IP1 and IP2,
  The lower bound for a is 10^{-10} and the upper bound is 10^3.

5. Results and discussion

We discuss, in this section, a couple of test examples to illustrate the accuracy and stability of the numerical results. For simplicity, we take h = T = 1. In addition, we investigate the problems for exact and noisy inputs data. We add noise to the measured input data (4) and (5) as

$$\mu_3^{noise}(t_j) = \mu_3(t_j) + random('Normal', 0, \sigma_1, 1, N), \quad j = 1, N \qquad (27)$$

$$\mu_4^{noise}(t_j) = \mu_4(t_j) + random('Normal', 0, \sigma_1, 0, N), \quad j = 0, N \qquad (28)$$

where the *random* is a command in MATLAB which generates random variables by the Gaussian normal distribution with zero mean and standard deviation σ_1 and σ_2, computed as

$$\sigma_1 = p \times \max|\mu_3(t)|, \ \sigma_2 = p \times \max|\mu_4(t)|, \ t \in [0,T]. \qquad (29)$$

where *p* is the percentage of noise. In order to analyse the error between the exact and the numerical results, a similar formula of equations (22) will be used.

5.1 Example 1 for (IP1)

Consider the IP1 given by (1)–(4) with unknown coefficient *a(t)* and solve this inverse problem with measured input data (18). One can observe that the plot of the function $\mu'_1(t) + \mu'_2(t) = (2 + e + \cosh(1))(3t^2 + 1)\exp(t^3 + t)$ does not vanish over the time interval [0,1] and hence the unique solvability guaranteed by Theorem 1. The analytical solution for this problem is given by equations (16) and (17) and it can be verified by direct substitution. Also, the direct problem (1)–(3) corresponding to current example has been previously solved numerically using FDM in Section 3.

Let us begin with the case of no noise contaminated in the input data (4). The naive objective function (unregularized); i.e, $\lambda = 0$, as a function of the number of iterations is plotted in Figure 3 for various mesh parameters. From this figure and Table 2 it can be notice that the unregularized objective function (25) decreases rapidly to a very low value of order about $O(10^{-24})$ in 7 iterations. The numerical solution for the corresponding coefficient a(t) is shown in Figure 4. From this figure, one can notice that as M = N increase, a better result we obtain, the rmse values decrease, indicating that we achieve mesh independence and excellent agreement are obtained.

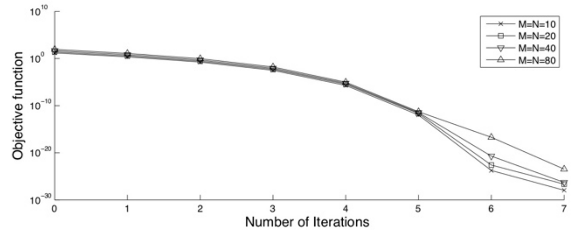

***Figure 3**. The unregularized objective function (25) with no noise, for various mesh size, for Example 1.*

Table 2. Number of iterations, number of function evaluations, value of objective function (25) at final iteration and the rmse values with no regularization and no noise for Example 1.

	M=N=10	M=N=20	M=N=40	M=N=80
No. of iteration	7	7	7	7
No. of function evaluations	96	176	336	656
Value of objective function (25) at final iteration	1.1E-28	2.2E-27	5.1E-27	3.4E-24
rmse(a)	0.2308	0.0548	0.0107	0.0024

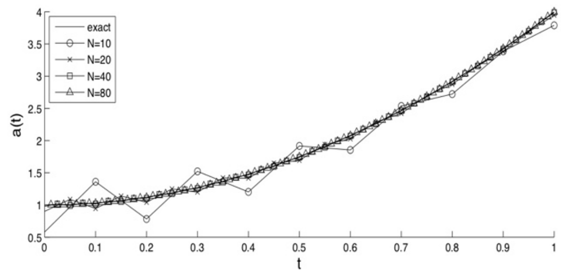

***Figure 4.** The coefficient a(t) for Example 1, with no noise and without regularization.*

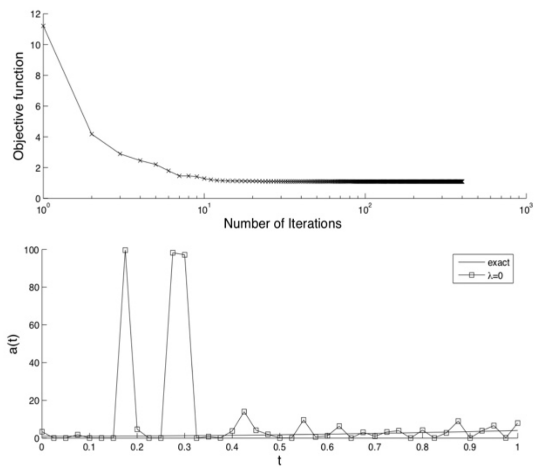

***Figure 5.** (up) the objective function (25) and (down) the coefficient a(t) for Example 1, with p=1% noise and without regularization.*

Next, let us fix the mesh parameter M = N = 40, for reasonable time consuming and add p = 1% noise to the measured input data $\mu_3(t)$ as given by (27). Figure 5 present the numerical results for the case of no regularization employed. From this figure it can be seen that the objective function (25) decreasing in slow manner whilst the unknown coefficient is unstable and unbounded solution. This behaviour is expected since the problem under investigation is ill-posed. Therefore, Tikhonov type regularization should be applied in order to retrieve the stability. The L-curve criteria [10] employed in order to choose an appropriate regularization parameter $\lambda > 0$ (for IP1) or $\beta > 0$ (for IP2), as shown in Figure 6(a), where the residual norm is $\|\int_0^1 u(x,t)dx - \mu_3(t)\|$.

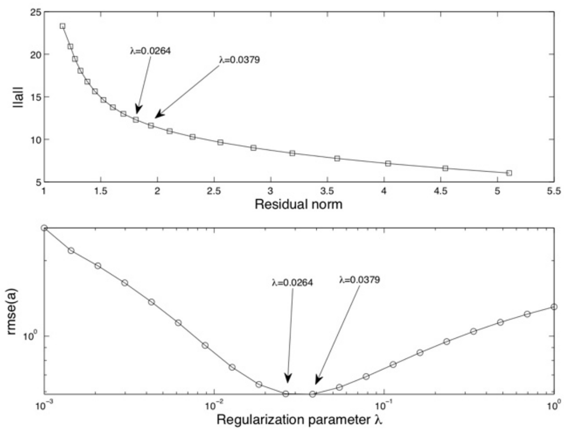

Figure 6. (up)The residual norm versus the solution norm for the L-curve with various regularization parameters, and (down) The regularization parameters versus the rmse vales for the coefficient a(t), for Example 1 with p = 1% noise.

Also, from this Figure 6, it can be seen that the two regularization parameter values located near the corner of the L-curve are $\lambda = \{0.0264, 0.0379\}$. These meet the minimum values of the *rmse* curve plotted versus regularization parameters in Figure 6(b). The associated numerical retrievals for the unknown a(t) when the two values of λ are selected are presented in Figure 7.

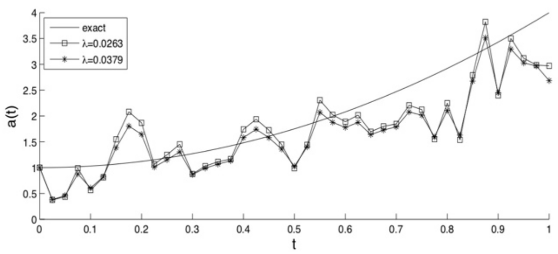

Figure 7. The numerical reconstructions for a(t) for Example 1, with p=1% noise.

From this figure, one can easily notice that as the regularization parameters $\lambda = 0.0379$ a better solution obtained see (-∗-) line. Moreover, the numerical solution for the temperature are depicted in Figure 8. From this figure, it can be seen that stable and accurate solutions are obtained. Also, it reported but not included that the same behaviour it can be seen for higher noise levels such as p = {3,5}%.

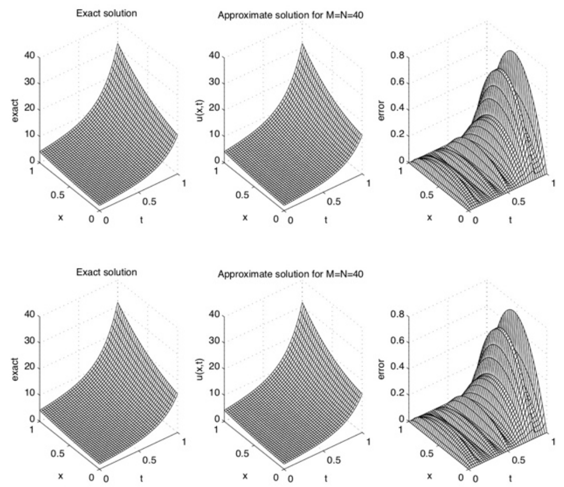

***Figure 8**. The exact and numerical solutions and the absolute error between them when (up) $\lambda = 0.0264$ and (down) $\lambda = 0.0379$, for Example 1, with p = 1% noise.*

5.2 Example 2 for (IP2)

Now, we consider the inverse problem (1)–(3) and (5) with the unknown coefficient a(t) and solve this problem with the same data as in Example 1, but notice the difference for equation (4), the integral type, replaced by (5), which is the difference of heat fluxes.

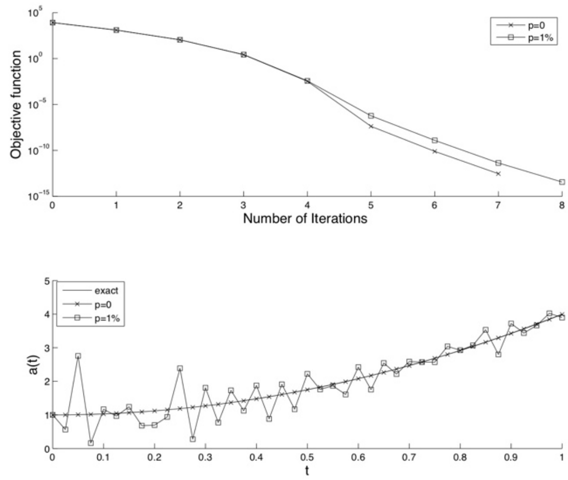

***Figure 9**. (up) The unregularized objective function (26), i.e, $\beta = 0$, and (down) The exact and numerical solution for a(t), for Example 2, with $p \in \{0,1\}$% level.*

As we notice that in previous example we can find the numerical solution for the temperature related with each noise percentage. Also, it is reported in this case we can obtain stable and accurate solutions when we apply regularization as we did in Example 1. Therefore, the results associated with regularization part has been omitted.

6. Conclusions

Two inverse coefficient identification problems have been numerically investigated. The finite difference method has been used in order to solve the direct problem. Whilst inverse problem recast as a nonlinear optimization problem which solved using MATLAB optimization toolbox. The numerical results are presented and it found accurate and stable.

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The Unique Maximal J-Regular Submodule

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Abstract: An R-module A is said to be J-regular module if, for each $a \in J(A)$, $r \in R$, there exist $t \in R$ such that $r\,a = r\,t\,r\,a$. We proved that each unitary R-module A contains a unique maximal J-regular submodule, which we denoted by M (A). Furthermore, the radical properties of A have investigated. We proved that if A is an R-module and N is a submodule of A, then J (N) ∩ M (A) ⊆ M (N). Moreover, if A is "projective," then M (A) = M (R) · J (A) and M (A) ∩ J(R).J (A) = (0).

Key Words: pure submodules**,** J-pure submodules, regular modules, J-regular modules.

Introduction

Throughout this paper, R is a commutative ring with identity and all modules are left, unitary, unless otherwise stated. An element $r \in$ R is said to be regular if there exists $t \in$ R such that $rtr = r$; a ring R is called regular if and only if each element of R is regular. An ideal I of a ring R is regular if each of its elements is regular in R; indeed, a regular ideal I of R is itself a regular ring [1]."Brown and McCoy proved in" [1] that each ring R contains a unique maximal regular ideal M(R), which satisfies the well-known radical properties. The ideal M(R) is called the regular radical of R. The concept of regularity extended to modules in several ways and in [2] the notion of F-regular modules (in the sense of Fieldhouse [3]) generalized to GF-regular modules . Let M be an R-module; an element $x \in$ M is, said to be GF-regular if for each $r \in$ R there exist $t \in$ R and a positive integer n such that $r^n t r^n x = r^n x$. An R-module M is called GF-regular if and only if all its elements are GF-regular. In [2] that each module contains a "unique maximal GF-regular submodule".
An R-module M is said to be J-regular module if for each $m \in J(M)$, $r \in R$, there exist $t \in R$ such that r m = r t r m [4].
A submodule N of an R- module A is called J- regular if each element of N is J- regular and every submodule of a J- regular module is a J- regular module. Also, the concept of J- pure submodule has been introduced. A submodule N of an R-module M is called a J-pure if N is pure in J (M), i.e. for each ideal I of R, I J (M) ∩ N = IN, where J (M) is the Jacobson radical of M [4]. In this paper, we show that each module contains a "unique maximal J- regular submodule," which we denoted by M (A), and we show that M (A) satisfies some but not all of the usual radical properties .

1. Main Results

Definition 1.1. Let A be an R -module. The unique maximal J-regular submodule of a module A denoted by M (A). If there exist a submodule containing every J-regular submodule of A, this means that M (A) is a J-"regular submodule" which is not contained properly in any J-"regular submodule".

Remarks and Examples 1.2
(1) If A = R, then M (A) is an ideal of R .
(2) It is clear that A is J- regular R- module if and only if M (A) = A.
(3) Since the Z-module Z_4 is J-"regular" [4]. Then M (Z_4) = Z_4.
(4) Each "submodule in the" Z -module Q is not J-regular, hence M (Q) = (0). Suppose that, M (Q) = B for some submodule B of Q implies that B is J-"regular" as Z-module. Take any element $x \in$ J (B), $x = \frac{a}{b}$ where a and b are two non – zero elements in Z. If we take an ideal <n> of Z where n is greater than one, then the non – zero cyclic submodule generated by $\frac{a}{b}$ is not J- pure in B, that is $<n>$ J (B) $\cap <\frac{a}{b}> \neq <n><\frac{a}{b}>$ which is a contradiction as B is J-regular.
(5) The Z– module Z is J- regular since J (Z) = 0 [4], then by remark 1.2 M (Z) = Z.
(6) The module Z_{P^∞} as Z –module is not J- regular. To show that, let $G_n = <\frac{1}{P^n} + Z>$ be any submodule of Z_{P^∞} where P is a fixed prime number and n is a positive integer . Then $\frac{1}{P^n} + Z = P^n\left(\frac{1}{P^{2n}} + Z\right) \in P^n J(Z_{P^\infty}) \cap G_n$, but $\frac{1}{P^n} Z \notin P^n < \frac{1}{P^n} + Z > Z = 0_{Z_{P^\infty}}$.
Thus M (Z_{P^∞}) $\neq Z_{P^\infty}$. Now, since every submodule $G_n = <\frac{1}{P^n} + Z>$ of Z_{P^∞} is isomorphic to the module Z_{P^∞} as Z –module, where $Z_P \subset Z_{P^2} \subset Z_{P^3} \subset \cdots \subset Z_{P^n} \subset \cdots$ .Thus M (Z_{P^∞}) $\cong Z_{P^n}$ for some integer n ≥ 0, this follows from the fact that $Z_{P^\infty} = \bigcup_{n\geq 0} Z_{P^n} = \sum_{n\geq 0} < \frac{1}{P^n} + Z >$.
(7) If B is a J- regular submodule of an R-module A, then B is not necessary be a J- pure submodule of A. For example, consider the module Z_8 as Z– module, let B = $< \bar{4} > = \{\bar{0}, \bar{4}\} \cong Z_2$ be J- regular module, but B is not J- pure submodule of Z_8, since if I= 2Z is an ideal of Z, then I$J(Z_8) \cap \{\bar{0}, \bar{4}\} = 2\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \cap \{\bar{0}, \bar{4}\} \not\subseteq \{\bar{0}\}$. I $\{\bar{0},\bar{4}\} = 2\{\bar{0},\bar{4}\} = \{\bar{0}\}$. Hence, the maximal J- regular submodule of Z_8 is M $(Z_8) = <\bar{2}> + <\bar{4}> = <\bar{2}>$.
(8) Let A be an R-module and $M^*(A)$ be the maximal regular submodule of A then it may be $M^*(A) \neq M(A)$. For example the module Z_8 as Z-module. It is easily to show that the regular submodules of Z_8 are $<\bar{0}>$ and $<\bar{4}>$ since $<\bar{4}> \cong Z_2$, thus $M^*(Z_8) = <\bar{4}>$. But $M(Z_8) = <\bar{2}>$, therefor, $M^*(Z_8) \neq M(Z_8)$.

Theorem 1.3. Every R-module contains a unique maximal J-regular submodule.

Proof: Let A be an R-module and $G = \{N: N$ is a J-regular submodule of $A\}$. Notice that as (0) is a J-regular submodule then G is a non-empty set. Let $\{Ni\}$ be an ascending chain in G and $B = \cup_{i\in\Lambda}$ Ni. Let $b \in$ J (B) then $b \in$ J $(\cup_{i\in\Lambda}$ Ni$) = \cup_{i\in\Lambda}$ J(Ni). In particular, if Ni , for each i = 1,2. To show $J(N_1 \cup N_2)=J(N_1) \cup J(N_2)$. Let $x \in J(N_1 \cup N_2)$ then R_x is small in $N_1 \cup N_2$ by [3].There exist a submodule K of $N_1 \cup N_2$, such that $N_2 = N_1 \cup N_2 = R_x + K$, then implies $x \in J(N_2)$. Hence $J(N_1 \cup N_2) \subseteq J(N_1) \cup J(N_2)$.

Conversely, assume that $y \in J(N_1) \cup J(N_2)$, then either $y \in J(N_1)$ or $y \in J(N_2)$. If $y \in J(N_2)$, then R_y a small in $N_2 = N_1 \cup N_2$. So we obtain is a small in $N_1 \cup N_2$. Hence $y \in J(N_1 \cup N_2)$. There exists $j\in \Lambda$ such that $b \in J(N_j)$, but N_j is a J- regular submodule, then for each $r \in R$, there exist $t \in R$ such that $rtrb = rb$ therefore b is a J- regular element in B which implies that B is a J-regular R-module. Now, by Zorn's Lemma, G contains a maximal element, which we call it $\mathcal{M}$. To prove the uniqueness of $\mathcal{M}$, assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ be two "maximal J-regular submodules" in A, then for any maximal ideal P of R each of $\mathcal{M}_1p$ and $\mathcal{M}_2p$ "is semisimple over" R_p [4]. Now, let $\mathcal{M}_1p \cap \mathcal{M}_2p = K_p$; then $K_p \subseteq \mathcal{M}_1p$ and $K_p \subseteq \mathcal{M}_2p$, thus $\mathcal{M}_1p = K_p + A_1p$ and $\mathcal{M}_2p = K_p + A_2p$, where A_1p and A_2p are two submodules of Ap [5]. Hence, $\mathcal{M}_1p + \mathcal{M}_2p = A_1p + K_p + A_2p$, but each of A_1p, A_2p, and K_p is a"semisimple submodule;" thus $\mathcal{M}_1p + \mathcal{M}_2p$ is a "semisimple submodule" which implies that $\mathcal{M}_1p + \mathcal{M}_2p$ is J-"regular" [5]. So $\mathcal{M}_1 + \mathcal{M}_2$ is a J-"regular submodule" [4]. Now, as of $\mathcal{M}_1$ and $\mathcal{M}_2$ "is a maximal J-regular submodule" and hence $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{M}_2 = \mathcal{M}_1$.

Proposition 1.4. Let A be an R-module and N a submodule of A, then J (N) ∩ M $(A) \subseteq$ M (N).

Proof: Let $x \in J(N)$ and $x \in M(A)$, thus for each $r \in R$, $rx = rtrx$ for some $t \in R$. Then x is J- regular element in N, which means that $x \in$ M (N).

Proposition 1.5. Let A_1 and A_2 be R-modules, then M $(A_1 \oplus A_2) \subseteq$ M $(A_1) \oplus$ M (A_2).

Proof: Let $c \in$ M $(A_1 \oplus A_2)$ and A = $A_1 \oplus A_2$,then $c = (a, b)$, where $a \in A_1$ and b $\in A_2$. Since c is J-regular element in A, then each of a and b is J- regular element, in A_1 and A_2, respectively. Which means that $a \in$ M (A_1) and $b \in$ M (A_2), hence $c \in$ M $(A_1) \oplus$M (A_2).

Recall that "the annihilator of an element x of an R-module A denoted by ann(x) is defined to be ann(x) = {r ∈ R: rx = 0} and the annihilator of A denoted by ann(A) is defined to be ann(A) = {r ∈ R: rx = 0 for every $x \in$ A}. Clearly, ann(x) and ann(A) are ideals of R, [6]." In [4] we prove that $\frac{R}{ann(x)}$ is the regular ring for each $x \in$ J (A) if and only if A is J-regular R-module. In fact if $\frac{R}{ann(J(A))}$ is a regular ring, then A is J- regular.

Proposition 1.6. Let A and A′ be R- modules, and $f : A \rightarrow$ A' be an R-homomorphism; then f(M(A)) ⊆ M(f(A)).

Proof: Let $a \in$ M (A), then a is J- regular element in A and $a \in$ J (A), which implies that $\frac{R}{ann(a)}$ is regular ring, for each $a \in$ J (A) [4]. But ann$(a) \subseteq$ ann$((a))$ and $f(a) \in f$(J (A)) ⊆ J (A'), hence exists φ: $\frac{R}{ann(a)} \rightarrow \frac{R}{ann(f(a))}$ define by φ (r +ann(a)) = r + ann(f(a).

Since $\frac{R}{ann(a)}$ is regular ring, for each a ∈ J (A), then by [7] $\frac{R}{ann(f(a))}$ is regular ring, hence f(A) is J- regular R-module [4] and f(a) ∈ M(f(A)). Thus f(M (A)) ⊆ M ((A)).

Proposition 1.7. Let A be a J- regular R-module, then M $(\frac{A}{M(A)})$ = (0).

Proof: Since A is a J- regular, then M (A) =A. Thus M $(\frac{A}{M(A)})$ = M $(\frac{A}{A})$ = (0).

Remark 1.8. For any R- module A, M $(\frac{A}{M(A)}) \neq (0)$ in general. For examples, the module Z_8 as Z-module M $(\frac{Z_8}{M(Z_8)})$ = M $(\frac{Z_8}{<\bar{2}>}) \cong$ M$(Z_2) = Z_2$. Thus $(\frac{Z_8}{M(Z_8)}) \neq$ (0).

Proposition 1.9. For each R-module A, M (R) · $A \subseteq$ M (A).

Proof: For each $a \in A$, let f_a: R → A be an R-homomorphism defined by $f_a(r) = ra$ for each $r \in R$, then by Proposition 1.6, f_a (M(R)) ⊆ M (A). On the other hand, M (R). A = $\sum f_a$(M(R)). Hence M (R) · $A \subseteq$ M (A).

Remark 1.10. The reverse inclusion M $(A) \subseteq$ M (R) · A, in Proposition 1.9 is not true in general. For example, the module Z_4 as Z-module where M $(Z_4) = Z_4 \nsubseteq$ M (Z) $Z_4 = Z(Z_4)$.

Let J (R) be the Jacobson radical of a ring R. Brown and McCoy proved [1] that M (R) ∩ J (R) = (0), where R is F- regular ring and for R is J-regular ring [4] M (R) ∩ J (R) = (0). However, this is not true for J- regular modules for example, if A = Z_4 as Z-module, then M (A) = Z_4 and J $(A) = \{\bar{0}, \bar{2}\}$ but M (A) ∩ J $(A) = \{\bar{0}, \bar{2}\} \neq (0)$.

Lemma 1.11. Let A be an R-module and N be a J- pure submodule of A. Let I be an ideal of R, then N=IN if and only if N ⊆ IJ(A).

Proof: Since N is J- pure submodule in A, then N ∩ IJ(A) = IN [4], for some ideal I of R. If N = IN, then N ∩ IJ(A) ⊆ N and hence N ⊆ IJ (A).

Conversely, if N ⊆ IJ (A), Then N ∩ IJ(A) = N, but N ∩ IJ(A) = IN; since N is J-pure submodule. Hence N = IN.

Recall that "(Nakayama`s Lemma) for an ideal I of R then I ⊆ J(R) if and only if for every finitely generated R- module M, if IM = M, implies M = $< 0 >$ [5].

Lemma 1.12. Let I be an ideal of a ring R contained in J(R) and let N be a finitely generated J- pure submodule of an R- module A with N ⊆ I J(A) implies that N = $< 0 >$.

Proof: By Lemma 1.11, we obtain N = IN and since I ⊆ J(R) we have N = $< 0 >$ by "Nakayama`s Lemma." If A is "F-regular R-module" and M (A) is "pure submodule" of A, then M(A) ∩ J(R).A = (0) [7]. For J- regular R-module we have the following:

Proposition 1.13. Let A be an R-module. If M (A) is a J- pure submodule of A, then M (A) ∩ J (R) ·J (A) = (0).

Proof: Let $x \in$ M (A) ∩ J (R) · J (A), then $x \in$ M (A) and $x \in$ J(R) · J (A). Let N be the cyclic submodule generated by x. It's clear that N ⊆ M (A), since M (A) is J- regular module; then N is J- pure in M (A). But M (A) is J- pure in A so N is J-pure in A [4]. On the other hand, $\mathrm{N} \subseteq \mathrm{J(R).J(A)}$, hence by Lemma 1.12, we have $\mathrm{N} = 0$. Which implies that M (A) ∩ J(R) · J (A) = (0).

Recall that "an R-module P is said to be a projective module if for any homomorphism f : P → B and for any epimorphism g: A → B; where A and B are two R-modules there exist a homomorphism h: P →A such that $f = g \circ$ h. [8]".

Recall that "(Dual Basis Lemma) an R- module A is projective module if and only if there exists a family of elements { x_i: i∈ Λ} ⊆ M and {f_i: i∈ Λ} ⊂ M^*= Hom (M;R) such that for any $x \in$ M, $f_i(x)$ = 0 for almost all i, (equivalently, $f_i(x) \neq 0$ only for a finite number of i∈ Λ and $x = \sum_{i\in\Lambda} x_i f_i(x)$, [8]".

Theorem 1.14. Let A be a projective R-module then

(1) M (A) = M (R) · J (A),

(2) M(A) is a J- pure submodule of A for each ideal I in R.

Proof: (1) Let $x \in$ M (A). $x \in$ J (A). Since A is projective R- module, then by "Dual Basis Lemma" there exists a family { x_i: i∈ Λ} of A and {f_i: i∈ Λ} ⊆ M^*= Hom (A, R) where $f_i(x) \neq 0$ only for finite number of i∈ Λ and $x = \sum_{i\in\Lambda} x_i f_i(x)$. But $f_i(x) \in$ M (R) by Proposition 1.6. Thus M (A) ⊆ M (R) ·J (A). We get the other direction of the inclusion by Proposition 1.9. Therefor M (A) = M (R) · J (A).

(2) Let I be an ideal of a ring R.

$\mathrm{M(A)} \cap \mathrm{IJ(A)} = \mathrm{M(R)J(A)} \cap \mathrm{IJ(A)} = (\mathrm{M(R)} \cap \mathrm{I})\mathrm{J(A)}$.

But $\mathrm{M(R)}$ is J- pure ideal in R, then $\mathrm{M(R)} \cap \mathrm{I} = \mathrm{IM(R)}$.

Hence $\mathrm{M(A)} \cap \mathrm{IJ(A)} = \mathrm{IM(R)J(A)} = \mathrm{IM(A)}$.

Recall that "if A is an R-module, then the trace of A is $\mathrm{tr}(A) = \sum_{f\in A^*} f(\mathrm{A})$, where A^*=Hom(A,R),[9]"

Proposition 1.15. Let A be a J- regular R-module. If tr (A) = R, then R is a J-regular.

Proof: Since A is J-regular, then M (A) = A, Remark (1.2) (2) and then M (A) = f (M (A)) ⊆ M(R) by Proposition (1.6) where $f \in A^*$ = Hom(A, R). Thus R = tr (A) = $\sum_{f\in A^*} f(\mathrm{A})$ ⊆ M (R) implies R = M (R). Therefor R is a J-regular .

Proposition 1.16. Let A be a finitely generated R-module and M (A) + J (A) = A, then A is J- regular .

Proof: Since A is finitely generated, then J (A) is small submodule of A, but M (A) +J (A) =A, therefore M (A) = A and hence A is J- regular.

Remark 1.17. For any R-module A, M (A) + J (A) ≠ A in general. For example, the module Z_8 as Z- module is not J-regular where M (Z_8) + J (Z_8) = $<\bar{2}>$ +$<\bar{2}>$ =$<\bar{2}> \neq Z_8$.

Recall that "a submodule N of an R- module A is called an essential submodule of A if for each submodule L of A with N∩ L =0 implies L= 0," [8]. We have the following:

Proposition 1.18. Let N be a submodule of an R- module A and J (N) be an essential submodule of A. If M (N) = 0, then M (A) = 0 .

Proof: Since J (N) ∩ M (A) ⊆ M (N) by Proposition 1.4. Then 0 = J (N) ∩ M (A). But J (N) is an essential submodule of A, "thus M (A) = 0.

Recall that a submodule K of an R-module A is said to be stable if $f(K) \subseteq K$ for each R-homomorphism,$f: K \longrightarrow A$" [10]. **Proposition 1.19.** For any R-module A, then M (A) is "stable submodule" of A.

Proof: Let $f \in Hom(\mathrm{M(A)}, \mathrm{A})$. By proposition 1.6, $f(\mathrm{M(M(A))}) \subseteq \mathrm{M(A)}$. But M (M (A)) = M (A) since M (A) is J- regular. Thus, $f(\mathrm{M(A)}) = f(\mathrm{M(M(A))}) \subseteq \mathrm{M(A)}$. Hence M (A) is "stable submodule."

Recall that "a non-zero submodule K of an R-module A is said to be dense in A if K generates A, that is $\mathrm{A} = \sum_{f\in Hom(\mathrm{K,A})} f(\mathrm{K})$" [11].

Proposition 1.20. Let A be an R-module and M (A) be a "dense submodule" in A, then A is J- regular module.

Proof: Since M (A) is "dense" in A, then $\mathrm{A} = \sum_{f\in Hom(\mathrm{M(A),A})} f(\mathrm{M(A)})$. But M (A) is "stable submodule" of A by the previous Proposition 1.19, thus $f(\mathrm{M(A)}) \subseteq \mathrm{M(A)}$

implies $\mathrm{A} = \sum_{f\in Hom(\mathrm{M(A),A})} f(\mathrm{M(A)}) \subseteq \mathrm{M(A)}$. Then $\mathrm{A} = \mathrm{M(A)}$ therefore A is J- regular .

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ON ESSENTIAL (T-SMALL) SUBMODULES

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***Abstract*:** Let M be an R-module and T be a submodule of M. A submodule K of M is called ET-small submodule of M(denoted by $K \ll_{ET} M$) , if for any essential submodule X of M such that T⊆K+X implies that T⊆X. We study this mentioned definition and we give many properties related with this type of submodules.

***Keywords*:**T-small submodule, T-maximal submodule, T-Radical submodule, ET-small submodule, ET-maximal submodule, ET- Radical submodule.

1. Introduction

Throughout this paper R is a commutative ring with identity and M a unitary R-module. A proper submodule N of M is called small ($N \ll M$), if for any submodule K of M ($K \leq M$) such that K + N = M implies that N = M. A submodule N of M is essential ($K \leq_e M$) if $K \cap L = 0$, then L= 0 , for every $L \leq M$ [1] . A submodule N of M is closed ($N \leq_c M$) if N has no proper essential extensions inside M that is, if the only solution of the relation $N \leq_e K \leq_e M$ is N=K[2]. The submodule N of M is called an essential- small ($N \ll_e M$)submodule of M , if for every essential submodule T of M such that M = N + T implies T = M [3].

In [4] the authors introduced the concept of small submodule with respect to an arbitrary submodul ,that a submodule K of M is called T-small in M , denoted by $K \ll T$ M , in case for any submodule X of M, such that T⊆ K+X, implies that T⊆X .

In this work we introduce essential T-small (ET-small) submodule, where an R-module M and T be a submodule of M. A submodule K of M is called ET-small submodule of M (denoted by $K \ll_{ET} M$), if for any essential submodule X of M such that T⊆K+X implies that T⊆X. In the first section , we give the fundamental properties of of ET-small submodules, Also we give many relations between ET-small submodule and other kinds of small submodules.
In the second section, we introduce essential T-maximal(ET-maximal) submodules and the essential T-radical (ET-radical) submodules of M denoted by $Rad_{ET}M$, We give the fundamental properties of this concepts.

2. Essential T-small submodule.

Definition2.1:Let M be an R-module and let T be a submodule of M .A submodule K of M is called ET-small submodule of M (denoted by $K \ll_{ET} M$), if for any essential submodule X of M such that T⊆K+X implies that T⊆X .

Remarks and Examples 2.2:

1. Consider Z_6 as Z-module .Let T={$\bar{0},\bar{3}$}, K={$\bar{0},\bar{2},\bar{4}$}.The only essential submodule of Z_6 is Z_6 if $T \subseteq K+Z_6$,then $T \subseteq Z_6$ .Thus$K \ll_{ET} Z_6$.

2.It is clear that Every T-small submodule of M is ET-small submodule of M but the converse is not true as for the following Consider Z_{24} as Z-module and Let T={$\bar{0}$, $\bar{8}$, $\overline{16}$ }, N=$8Z_{24}$ the only essential submodule in Z_{24} are $2Z_{24}$, $4Z_{24}$ and Z_{24} , T=$8Z_{24} \subseteq 8Z_{24} + 2Z_{24}$ and $8Z_{24} \subseteq 2Z_{24}$,also $8Z_{24} \subseteq 8Z_{24} + 4Z_{24}$, $8Z_{24} \subseteq 4Z_{24}$ and $8Z_{24} \subseteq Z_{24}$ Then $8Z_{24}$ ET-small submodule of Z_{24} which is not T-small submodule of Z_{24} since $8Z_{24} \subseteq 8Z_{24}+3Z_{24}but8Z_{24} \nsubseteq 3Z_{24}$.

3. Let M be an R- module and T=0.Then every essential submodule of M is ET-small in M.

4. Let M be an R-module and T=M .Then $N \ll_{ET} M$ if and only if $N \ll_e M$.

Proposition 2.3: Let M be an R-module and let T,H and L be submodules of M such that T ≤ N and H ≤ N ≤ M and $N \ll_e M$. If $H \ll_{ET} M$, then $H \ll_{ET} N$.

Proof: Let H be ET-submodules of M and X be an essential submodule of N such that T⊆H+X .since $X \leq_e N$ and $N \leq_e M$ so $X \leq_e M$[2], then $H \ll_{ET} M$, and T⊆X .

Proposition 2.4: Let M be an R-module with submodules N ≤ H ≤ M such that T≤ H .If $N \ll_{ET} H$, then $N \ll_{ET} M$.

Proof*:* Suppose that T⊆N+X, for any essential submodule X of M .Since T⊆H, then T=T⋂H⊆(N+X)⋂H=N+(X⋂H) by modular law , since $X \leq_e M$ and $H \leq_e H$, then $(X \cap H) \leq_e (M \cap H)$ =H [2], and $N \ll_{ET} H$, then T⊆X. Thus $N \ll_{ET} M$.

Proposition 2.5: Let M be an R-module and Let T, N_1 and N_2 be a submodules of M, Then $N_1 \ll_{ET} M$ and $N_2 \ll_{ET} M$ if and only if $N_1+N_2 \ll_{ET} M$.

Proof*:* Suppose that $N_1 \ll_{ET} M$ and $N_2 \ll_{ET} M$ and Let $T \subseteq (N_1+N_2)+X$, for any essential submodule X of M, then $T \subseteq N_1+(N_2+X)$, since $X \leq_e M$ and $N_2 \leq_e N_2$, then $N_2+X \leq_e M$ [2] and $N_1 \ll_{ET} M$ Then $T \subseteq N_2+X$, since $N_2 \ll_{ET} M$ Then T⊆ X. Conversely, let $N_1+N_2 \ll_{ET} M$, to show that $N_1 \ll_{ET} M$ and $N_2 \ll_{ET} M$, Suppose that $T \subseteq N_1 +X$, for any essential submodule X of M , since $N_1 \subseteq N_1+N_2$,so $T \subseteq N_1+N_2+X$, but $N_1+N_2 \ll_{ET} M$, so T⊆ X, thus $N_1 \ll_{ET} M$, and the same we have $N_2 \ll_{ET} M$.

Proposition 2.6: Let M be an R-module and Let H be a submodule of M .If $\{T_i\}_{i \in I}$ be a family set of submodules of M such that $H \ll_{ET_i} M$, for each $i \in I$, then $H \ll_{E(\sum_{i \in I} Ti)}$ M .

Proof:Let ($\sum_{i \in I} T_i$)⊆ H+X, for any essential submodule X of M. then for each $i \in I$, $T_i \subseteq H+X$ and by hypothesis $T_i \subseteq X$, thus ($\sum_{i \in I} T_i$)⊆ X.

Proposition2.7: Let M and N be any R-modules and f : M → N be a homomorphism .If T and H are submodules of M such that $H \ll_{ET} M$,then $f(H) \ll_{Ef(T)} N$.

Proof: Let f (T) ≠ 0 and f (T)⊆f (H)+X , for any essential submodule X of N .to show T⊆H+ f^{-1}(X) . let $t \in T$, then $t=h+w$,for some $h \in H$ and $b \in f^{-1}$(X) .Hence $f(t)=f(h+b)=f(h)+f(b)$ .Thus $f(t)$-$f(h) = f(b)$, thus $f(t-h) = f(b) \in X$ and so($t-h) \in f^{-1}$(X) implies that $t \in H+ f^{-}$

^{1}(X).since $X \leq_e N$ Thus $f^{-1}(X) \leq_e M$ [2] and $H \ll_{ET} M$,therefore $T \subseteq f^{-1}(X)$ .Thus $f(T) \subseteq X$.

Proposition 2.8: Let M be an R-module and Let T, H and N be submodules of M such that $H \leq N \leq M$ and $H \leq T$. if $N \ll_{ET} M$ then $H \ll_{ET} M$ and $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$.

Proof: Let $N \ll E_{GT}$ M .To show that $H \ll_{ET} M$, let $T \subseteq H+X$, for any essential submodule X of M .but $H \leq N \leq M$, so $T \subseteq N+X$ .then $T \subseteq X$ .Thus $H \ll_{ET} M$. Now to show that $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$, let $\frac{T}{H} \subseteq \frac{N}{H} + \frac{X}{H}$, for any essential submodule $\frac{X}{H}$ of $\frac{M}{H}$ such that $H \subseteq X$. Then $\frac{T}{H} \subseteq \frac{N+X}{H}$ so $T \subseteq N+X$, Since $\frac{X}{H} \leq_e \frac{M}{H}$ then $X \leq_e M$ [3] , and $N \ll_{ET}$ M, then $T \subseteq X$ Thus $\frac{T}{H} \subseteq \frac{X}{H}$.

Proposition2.9: Let M be an R-module and Let T, H and N be submodules of M such that $H \leq N \leq M$ and $H \leq T$ and $H \leq_c M$, if $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$ then $N \ll_{ET} M$.

Proof: Let $H \ll_{ET} M$ and $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$,to show that $N \ll_{ET}$ M, let $T \subseteq N+X$, for any essential submodule X of M and $H \subseteq X$. Now $\frac{T}{H} \subseteq \frac{N+X}{H} = \frac{N}{H} + \frac{X}{H}$, since $X \leq_e M$ and $H \leq_c M$ then $\frac{X}{H} \leq_e \frac{M}{H}$ [3]. But $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$, so $\frac{T}{H} \subseteq \frac{X}{H}$ implies that $T \subseteq X$ .thus $N \ll_{ET} M$.

Corollary2.10: Let M be an R-module and Let K and H be submodules of M such that $K \ll_{EH} M$ and $H \ll_{EK} M$. Then $(H \cap K) \ll_{E(K+H)} M$.

Proof: Let $K \ll_{EH} M$ and $H \ll_{EK} M$, since $(H \cap K) \leq H$ and $(H \cap K) \leq K$, by Proposition (1.4),$(H \cap K) \ll_{EK} M$ and $(H \cap K) \ll_{EH} M$. Also by Proposition (1.6) we get $(H \cap K) \ll_{E(K+H)} M$.

Proposition 2.11: Let M be an R-module and Let T , K ,H and B be submodules of M such that $K \leq H \leq B \leq M$, $K \leq_c M$ and $H \leq_c M$. Then $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ if and only if $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$ and $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$.

Proof: Let $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$.To show that $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$, let $\frac{T+H}{H} \subseteq \frac{B}{H} + \frac{X}{H} = \frac{B+X}{H}$,for any essential submodule $\frac{X}{H}$ of $\frac{M}{H}$ and $H \subseteq X$. Hence $T \subseteq T+H \subseteq B+X$ .So $\frac{T+K}{K} \subseteq \frac{B+X}{K}$ .Therefore $\frac{T+K}{K} \subseteq \frac{B}{K} + \frac{X}{K}$ .since $X \leq_e M$ and $K \leq_c M$ then $\frac{X}{k} \leq_e \frac{M}{k}$ [3] But $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$,therefore $\frac{T+K}{K} \subseteq \frac{X}{K}$ .So $T \subseteq T+K \subseteq X$ and hence $\frac{T+H}{H} \subseteq \frac{X}{H}$ .Thus $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$. To show that $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .Let $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K}$,for any essential submodule $\frac{X}{K}$ of $\frac{M}{K}$, $K \subseteq X$. Then $\frac{T+K}{K} \subseteq \frac{H+X}{K}$ and hence $T \subseteq T+K \subseteq H+X \subseteq B+X$ implies that $\frac{T+K}{K} \subseteq \frac{B+X}{K} = \frac{B}{K} + \frac{X}{K}$, Since $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$, then $\frac{T+K}{K} \subseteq \frac{X}{K}$ .Thus $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .Conversely, let $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$ *and* $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .To shows that $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ .Let $\frac{T+K}{K} \subseteq \frac{B}{K} + \frac{X}{K}$, for any essential submodule $\frac{X}{K}$ of $\frac{M}{K}$.Then $\frac{T+K}{K} \subseteq \frac{B+X}{K}$ and hence $T \subseteq T+K \subseteq B+X$ .so $T+H \subseteq B+X+H$ implies that $\frac{T+H}{H} \subseteq \frac{B+X+H}{H} = \frac{B+X}{H}$. therefore $\frac{T+H}{H} \subseteq \frac{B}{H} + \frac{X}{H}$, since $X \leq_e M$ and $H \leq_c M$ then $\frac{X}{H} \leq_e \frac{M}{H}$ [3] and $\frac{B}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$, therefore $\frac{T+H}{H} \subseteq \frac{X}{H}$. Then $T+H \subseteq X$ But $T+k \subseteq T+H$ and $X \subseteq X+H$, so $T+k \subseteq T+H \subseteq X \subseteq X+H$ therefore $T+k \subseteq X+H$, so $\frac{T+K}{K} \subseteq \frac{X+H}{K} = \frac{X}{K} + \frac{H}{K}$, since $X \leq_e M$ and $K \leq_c M$ then $\frac{X}{K} \leq_e \frac{M}{K}$ and $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ then $\frac{T+K}{K} \subseteq \frac{X}{K}$.Thus $\frac{B}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$.

Proposition2.12: Let $M=M_1 \oplus M_2$ be an R-module such that $R=Ann(M_1)+Ann(M_2)$ .If $N_1 \ll_{ET1} M_1$ and $N_2 \ll_{ET2} M_2$, then $N_1 \oplus N_2 \ll_{E(T1 \oplus T2)} M$.

Proof: Let $T_1 \oplus T_2 \subseteq N_1 \oplus N_2+X$, for any essential submodule X of M .Since $R=Ann(M_1)+Ann(M_2)$ .Then, by the same argument of the prove of [6, prop. 4.2, ch 1] $X= X_1 \oplus X_2$,for any essential submodule X_1 of M_1 and submodule X_2 of M_2 .Hence $T_1 \oplus T_2 \subseteq N_1 \oplus N_2+X_1 \oplus X_2$,implies that $T_1 \oplus T_2 \subseteq (N_1+X_1) \oplus (N_2+X_2)$ .to show that $T_1 \subseteq N_1+X_1$ and $T_2 \subseteq N_2+X_2$ .let $t_1 \in T_1$and $t_2 \in T_2$ then $t_1+t_2 \in T_1 \oplus T_2 \subseteq (N_1+X_1) \oplus (N_2+X_2)$, so $t_1 \in (N_1+X_1)$ and $t_1 \in (N_2+X_2)$, then $T_1 \subseteq N_1+X_1$ and $T_2 \subseteq N_2+X_2$, Since $N_1 \ll_{ET1} M_1$ and $N_2 \ll_{ET2} M_2$, then $T_1 \subseteq X_1$ and $T_2 \subseteq X_2$ and hence $T_1 \oplus T_2 \subseteq X_1 \oplus X_2= X$ .Thus $N_1 \oplus N_2 \ll_{E(T1 \oplus T2)} M$.

Recall that an R-module M is called a fully stable module if for each submodule K of M and for each R-homomorphism f from M into K, $f(K) \subseteq K$ [5].

Proposition2.13: Let $M=\oplus_{i \in I} M_i$ be a fully stable module .If $K_i \ll_{E(Ti)} M_i$, for each $i \in I$, then $\oplus_{i \in I} K_i \ll_{E(\oplus_{i \in I} T_i)} \oplus_{i \in I} M_i$.

Proof: Let $M=\oplus_{i \in I} M_i$ be a fully stable module and $K_i \ll_{E(Ti)} M_i$, for each $i \in I$ .To show that $\oplus_{i \in I} K_i \ll_{E(\oplus_{i \in I} T_i)} \oplus_{i \in I} M_i$ .Let $(\oplus_{i \in I} T_i) \subseteq (\oplus_{i \in I} K_i)+X$, for any essential submodule X of M .Claim that $X=\oplus_{i \in I}(X \cap M_i)$ .To show that, for each $i \in I$ let $Pi : M \to M_i$ be The projection map and let $x \in X$,then $x \in \oplus_{i \in I} M_i$ and hence $x=\sum_{i \in I} xi$ where $xi \in Mi$, $\forall i \in I$ and $xi \neq 0$ for at most a finite number of $i \in I$ .Since M is fully stable, then $Pi(x) \in X$, $\forall i \in I$ .Now $P_i(x) = P_i(\sum_{i \in I} x_i) = x_i \in (X \cap Mi)$ and hence $x= (\sum_{i \in I} xi) \in \oplus_{i \in I}(X \cap M_i)$ .Thus $X \subseteq \oplus_{i \in I}(X \cap M_i)$ .Clearly $\oplus_{i \in I}(X \cap M_i) \subseteq X$ .Thus $K= \oplus_{i \in I}(K \cap M_i)$ .Now $\oplus_{i \in I} T \subseteq (\oplus_{i \in I} K_i)+(\oplus_{i \in I}(X \cap M_i)) = \oplus_{i \in I}(K_i+(X \cap M_i))$ .Therefore $T_i \subseteq K_i+(X \cap M_i)$,for each $i \in I$ .Since $K_i \ll_{E(Ti)} M_i$,then $Ti \subseteq (X \cap Mi)$ and hence $\oplus_{i \in I} T \subseteq \oplus_{i \in I}(X \cap Mi)=X$. thus $\oplus_{i \in I} K_i \ll_{E(\oplus_{i \in I} T_i)} \oplus_{i \in I} M_i$.

Recall that the annihilator of M $Ann(M) =\{r \in R \mid rM = 0\}$ [6], M is a faithful module if $Ann(M) = 0$. M is a multiplication module if for each submodule N of M, there exists an ideal I of R such that N=IM [7].

Proposition2.14: Let M be a finitely generated, faithful and multiplication module and let I, J be ideals in R .Then $I \ll_{EJ} R$ if and only if $IM \ll_{E(JM)} M$.

Proof: Let $I \ll_{EJ} R$ .To show that $IM \ll_{E(JM)} M$ .Let $JM \subseteq IM+X$, for any essential submodule X of M .Since M is multiplication module, then X=KM, for some ideal K of R and hence $JM \subseteq IM+KM= (I+K)M$ .since M be a finitely generated, faithful and multiplication module, therefore M is a cancellation module, by [9] .then $J \subseteq I+K$ since $K \leq_e R$ .Since $I \ll_{EJ} R$,then $J \subseteq K$ .Hence $JM \subseteq KM=X$ .Thus $IM \ll_{E(JM)} M$. Conversely, let $IM \ll_{E(JM)} M$ .To show that $I \ll_{EJ} R$ .Let K be essential ideal of R such that $J \subseteq I+K$ .Since M is multiplication module, then $JM \subseteq IM+KM$ .But $IM \ll_{E(JM)} M$, therefore $JM \subseteq KM$ .So $J \subseteq K$ .Thus $I \ll_{EJ} R$.

3. Essential T-Radical of M. Recall that if M an R-module and T be a submodule of M. A submodule K of M is called T-maximal submodule of M if is $\frac{T+K}{K}$ simple [4]. In this section, we introduce the definitions of ET-maximal submodules and ET-radical of a module as a generalization of T-maximal submodules and T-radical of a module and we discuss some of the basic properties of this concepts.

Definition 3.1: Let M be an R-module and let T be a submodule of M .An essential submodule K of M is called essential T-maximal(ET-maximal) submodule of M if $\frac{T+K}{K}$ is simple.

Remarks and examples 3.2:

1.If M is a unform R-module M and let K be a submodule of a module M, then K is ET-maximal submodule of M if and only if K is T- maximal submodule of M.

2.If T a submodule of M then every ET-maximal submodule of M is T- maximal submodule of M but the converse is not true as the following example.

Consider Z_6 as Z-module .Let $T=\{\bar{0},\bar{2},\bar{4}\}$ and $K=\{\bar{0},\bar{3}\}$.Then K is T-maximal submodule of Z_6 , where $\frac{\{\bar{0},\bar{2},\bar{4}\}+\{\bar{0},\bar{3}\}}{\{\bar{0},\bar{3}\}} = \frac{Z_6}{\{\bar{0},\bar{3}\}} \cong \{\bar{0},\bar{2},\bar{4}\}$ is simple, but K is not ET-maximal submodule of Z_6,since K is not essential submodule of Z_6.

Since every ET-maximal submodule of M is T-maximal submodule .The following we get without prove since the prove is as the same way on [4],[8]

Proposition 3.3: Let M be an R-module and($0 \neq T$) be a proper finitely generated submodule of M and let

$A = \{L \le M \mid L \ll_{ET} M$ and $L+K \subseteq T+K$, for all ET-maximal submodule K of M$\}$ and

$B = \{K \le M \mid K$ is an ET-maximal submodule of M$\}$. Then $\sum_{L\in A} L = \bigcap_{K\in B} K$.

Proposition 3.4: Let M be an R-module and be a finitely generated submodule of M and $a \in M$ Then Ra is not ET-small submodule of M if and only if there exists H is ET-maximal submodule of M such that $a \notin H$ and $T \subseteq Ra + H$.

Proposition 3.5: Let M and N be an R-modules and f : M → N be an R-homomorphism. If T is a submodule of M and K is an ET-maximal submodule of M such that kerf ⊆ K, then f(K) also is an Ef(T)-maximal submodule of N.

Proposition 3.6: Let M and N be an R-modules and f : M → N be an R- epimorphism . If T is a submodule of M and K is an Ef(T)-maximal submodule of N, then $f^{-1}(K)$ also is an ET-maximal submodule of M.

Proposition 3.7: Let H and T be submodules of a module M such that T is finitely generated and $T \nsubseteq H$ .Then there exists a ET-maximal submodule of M containing H.

Definition 3.8: Let M be an R – module the intersection of all essential T-maximal submodules of M is called a essential T-Radical of M (denoted by $Rad_{ET}(M)$) .If M has no ET-maximal submodule , then $Rad_{ET}(M) = T$.

Remarks and Examples 3.9:

1. If M be an uniform R-module then $Rad_{ET}(M) = Rad_T(M)$.

2.If.T=M.then.$Rad_{ET}(M)=Rad_e(M)$.

3.Consider Z_6 as Z-module .Let $T=Z_6$ and $K_1 = Z_6$ are ET-maximal submodules of Z_6 , therefore $Rad_{GT}(Z_6) = Z_6$.

4. Consider Z_4 as Z-module .Let $T= Z_4$ and $K= \{\bar{0},\bar{2}\}$, then K is the only ET-maximal submodule of Z_4 .To show that, $\frac{Z_4+\{\bar{0},\bar{2}\}}{\{\bar{0},\bar{2}\}} \cong \{\bar{0},\bar{2}\}$ is a simple .Thus $Rad_{ET} Z_4 = Rad_T Z_4 = \{\bar{0},\bar{2}\}$.

5.Consider Z_{p^∞} as Z-module .Let $T= Z_{p^\infty}$, then Z_{p^∞} has no ET-maximal submodule and hence $Rad_{ET} Z_{p^\infty} = Z_{p^\infty}$.

Proposition 3.10: Let M be an R-module and let T be a finitely generated submodule of a module M .Then $Rad_{ET}(M) \ll_{ET} M$.

Proof: Assume that $T \subseteq Rad_{ET}M + X$, for any essential submodule X of M .to show that $T \subseteq X$ suppose that $T \nsubseteq X$ .Then by *Proposition* (2.7) , there exists a ET-maximal submodule K of M such that $X \subseteq K$ .Therefore $T \subseteq Rad_{ET}M + X \subseteq K$ .implies that $T \subseteq K$, so $\frac{T+K}{K} = 0$ which contradicts the T-maximality of K. Thus $T \subseteq X$, Thus $Rad_{ET}(M) \ll_{ET} M$.

Lemma 3.11: Let M be an R- module and let T be a finitely generated submodule of a module M and $m \in M$ such that $R_m+H \subseteq T+H$, for all ET-maximal submodule H of M, then $R_m \ll_{ET} M$ iff $m \in Rad_{ET}(M)$.

Proof: Let $R_m \ll_{ET} M$ and $R_m + H \subseteq T+H$, for all ET-maximal submodule H of M , By Proposition (2.3) then $R_m \in A$,where $A = \{L \le M \mid L \ll_{ET} M$ and $L+H \subseteq T+H$, for all ET-maximal submodule H of M$\}$. Hence $R_m \subseteq Rad_{ET}M$. For the converse, let $m \in Rad_{ET}M$. To show that $R_m \ll_{ET} M$. Suppose that R_m is not ET-small submodule M . By Proposition (2.4), then there exists H is a ET-maximal submodule of M with $m \notin H$ then $m \notin Rad_{ET}M$ which is a contradiction .Thus R_m is a ET-small submodue of M.

Proposition 3.12: Let M and N be an R-modules and f : M → N be an R-epimorphism such that $Kerf \subseteq Rad_{ET}M$. Then $f(Rad_{ET}M) = Rad_{Ef(T)}N$.

Proof: Since f is epimorphism, by *Proposition* (2.5) and Proposition(2.6),we,have $f(Rad_{ET}M) = f(\bigcap_{K\in A} K) = \bigcap_{f(k)\in B} f(K) = Rad_{Ef(T)}N$, where, $A = \{K \le M \mid K$ is an ET-maximal submodule of M$\}$ and $B=\{f(K) \le N \mid f(K)$ is an Ef(T)-maximal submodule of N$\}$.

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ON A CLASS OF ANALYTIC MULTIVALENT FUNCTIONS INVOLVING HIGHER-ORDER DERIVATIVES

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Abstract

In this paper, we get some interesting geometric concepts of the class of multivalent functions involving higher order derivatives defined on the open unite disk U. We obtain some interesting properties, like , coefficient inequalities, distortion and growth property , closure property, radius of stalikness and radius of convexity and hadamard product .

Keywords: Analytic , Multivalent , higher order derivatives.

1.Introduction.

Let $S_p(n)$ denoted the class of analytic functions:
$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $(p, n \in \mathrm{N} = \{1, 2, 3, \dots\})$ ... (1.1), are p−valent in unit disc U = { {z : z ∈ ℂ and |z| < 1}. Let $T_p(n)$ denoted the subclass of Sp(n) of the following form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0) \dots\dots (1.2)$$

We note that $T_p(1) = T_p$.

For all $(z) \in S_p(n)$, we have
$f^{(m)}(z) = \delta(p,m) z^{p-m} + \sum_{k=p+n}^{\infty} \delta(k,m) a_k z^{k-m}$ (1.3)

Where

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} i(i-1)(i-2)\dots\dots(i-j+1) & j \neq 0 \\ 1 & j = 0 \end{cases} \dots\dots (1.4)$$

Aouf [1] introduced and studied the class $T_p^*(\lambda, l, \alpha, \beta)$ consisting of functions $f(z) \in S_p(n)$ which satisfies:

$$\left| \frac{A\left\{\frac{f'(z)}{z^{p-1}} - p(p-1)\right\}}{B\left\{\frac{f'(z)}{z^{p-1}} - p(p-1)\right\} + \lambda(1-\alpha)} \right| < (l-\beta) \dots\dots\dots (1.5)$$

Where $0 < B \leq 1 , A \geq 0 , \lambda > 0 , 0 \leq \alpha < 1 , 0 < l < B < 1 , p \in N \text{ and } z \in U$.

Let $S_n\ (p, q; , A, B, \lambda, \alpha, l, \beta)$be the subclass of $S_p(n)$ consisting of functions $f(z)$ of the form (1.1), and satisfying the analytic criterion:

$$\left| \frac{A\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q) z^{p-q-2}} - p(p-1)\right\}}{B\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q) z^{p-q-2}} - p(p-1)\right\} + \lambda(1-\alpha)} \right| < (l-\beta) \dots\dots\dots (1.6)$$

Where $0 < B \leq 1 , A \geq 0 , \lambda > 0 , 0 \leq \alpha < 1 , 0 < l < B < 1, p \in N_0 = N \cup \{0\}$ and $p > q$

Further, let
$T_n^*\ (p, q; A, B, \lambda, \alpha, l, \beta) = S^*\ (p, q;\ A, B, \lambda, \alpha, l, \beta) \cap T_p(n)$ .... (1.7)

For suitable choices of $n, p, q, A, B, \lambda, l, \alpha \text{ and } \beta$ we obtain the following subclasses:

(i) $T_1^*(p, 0; , A, B, \lambda, \alpha, l, \beta) = P_p^*(A, B, \lambda, \alpha, l, \beta)$ $(Aouf[1])$;

(ii) $T_1^*(1, 0; A, B, \lambda, \alpha, l, \beta) = P^*(A, B, \lambda, \alpha, l, \beta)$ $(Gupta\ and\ Jain[2])$

(iii) $T_1^*(p, 0; , A, B, \lambda, \alpha, l, 1) = F_p(A, B, \lambda, 1, l, \alpha)$ $(Lee\ et\ al[3])$

Also , we note that :
$T_n^*\ (p, q; A, B, \lambda, \alpha, l, 1) = T_n^*(p, q; A, B, \lambda, \alpha, l) = \left\{ f \in T_p(n): Re\left(\frac{f^{(q+2)}(z)}{\delta(p-2,q) z^{p-q-2}}\right) > \alpha, 0 \leq \alpha < p \right\}$

2.Coefficient inequalities

We assume throughout this paper that $0 < B \leq 1 , A \geq 0 , \lambda > 0 , 0 \leq \alpha < 1 , 0 < l < B < 1, p \in N_0 = N \cup \{0\}$ and $p > q$ and $\delta(i,j)(i > j)$ is defined by (1.4).

theorem 1. A function $f(z)$ of the form (1.2) is in the class $T_n^*\ (p, q, A, B, \lambda, \alpha, l, \beta)$ if and only if
$\sum_{k=p+n}^{\infty} [A + B(l-\beta)] k(k-1) \delta(k-2,q) a_k \leq \lambda(l-\beta)(1-\alpha)\delta(p-2,q)$…..(2.1)

Proof. Assume that the inequality (2.1) holds true , then
$\left|A\left\{f^{(q+2)}(z) - p(p-1)\delta(p-2,q)\ z^{p-q-2}\right\}\right| - (l-\beta)\left|B\left\{f^{(q+2)}(z) - p(p-1)\ \delta(p-2,q) z^{p-q-2}\right\} + \lambda(1-\alpha)\delta(p-2,q)\ z^{p-q-2}\right|$

$=\left|A\left\{\delta(p,q+2)\,z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_k\,z^{k-q-2}-\,p(p-1)\,\delta(p-2,q)\,z^{p-q-2}\right\}\right|-(l-\beta)\left|B\left\{\delta(p,q+\;2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)\,a_k\,z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}\right|$

We have $\delta(p,q+2)=p(p-1)\delta(p-2,q)$ then

$=\left|A\left\{p(p-1)\delta(p-2,q)z^{p-q-2}+\sum_{k=p+n}^{\infty}k(k-1,q)\delta(p-2,q)a_kz^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}\right|-(l-\beta)\left|B\left\{p(p-1)\delta(p-2,q)z^{p-q-2}+\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}\right|$

then

$=\left|A\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}\right|-(l-\beta)\left|B\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}\right|$

$$\le\sum_{k=p+n}^{\infty}[A+B(l-\beta)](k(k-1)\delta(k-2,q)a_k|z|^{k-q-2}-\lambda(1-\alpha)(l-\beta)\delta(p-2,q)|z|^{k-q-2}$$

$=\sum_{k=p+n}^{\infty}[A+B(l-\beta)](k(k-1)\delta(k-2,q)a_k\le\lambda(1-\alpha)(l-\beta)\delta(p-2,q)$

Conversely , assume that $f(z)\in T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ thus

$$\left|\frac{A\left[f^{(q+2)}(z)-p(p-1)\delta(p-2,q)z^{p-q-2}\right]}{B\left\{f^{(q+2)}(z)-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}}\right|$$

$$=\left|\frac{A\left[\delta(p,q+2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_kz^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right]}{B\left[\delta(p,q+2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_kz^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right]+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}}\right|<l-\beta\;.$$

We have $\delta(p,q+2)=p(p-1)\delta(p-2,q)$ then

$$\left|\frac{A\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}}{B\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}}\right|<l-\beta$$

Since $Re(z)\le|z|$ for all z , we get

$$Re\left|\frac{A\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}}{B\sum_{k=p+n}^{\infty}k(k-1)\delta(k-2,q)a_kz^{k-q-2}+\lambda(1-\alpha)\delta(p-2,q)\,z^{p-q-2}}\right|<l-\beta\;\ldots\ldots\ldots(2.2)$$

Taking values of z on the real axis then $\frac{f^{(q+2)}(z)}{\delta(p-2,q)z^{p-q-2}}$ is real then , upon cleaning the denominator in (2.2) and putting $z\to-1$, we get the desired result .

Corollary 1. Let the function $f(z)$defined by (1.1) be in the class $T_n^*\;(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then

$$a_k\le\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}$$

Sharpness is hold for

$$f(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k,$$

$(k\ge n+p\;,n\in N)$

3.Distortion property

theorem 2. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\;(p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then $|z|=r<1$ we have

$$\left(\delta(p,m)-\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}r^m\right)r^{p-m}\le\left|f^{(m)}(z)\right|\le\left\{\delta(p,m)+\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}r^m\right\}r^{p-m}\;\ldots..(3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}z^{p+n}\;\ldots\ldots(3.2)$$

Proof . By theorem 1 , we have

$$[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)\sum_{k=p+n}^{\infty}a_k\le\lambda(l-\beta)(1-\alpha)\delta(p-2,q)$$

$$\le\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)a_k\le\lambda(l-\beta)(1-\alpha)\delta(p-2,q)\;\ldots\ldots(3.3)$$

That is

$$\sum_{k=p+n}^{\infty}a_k\le\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\;\ldots\ldots(3.4)$$

From (1.3) and (3.4)

$$\left|f^{(m)}(z)\right|\ge\left\{\delta(p,m)r^{p-m}-r^{p+n-m}\delta(p+n,m)\sum_{k=p+n}^{\infty}a_k\right\}$$

$$\ge\left\{\delta(p,m)r^{p-m}-r^{p+n-m}\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}\right\}$$

$$\ge\left\{\delta(p,m)-\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}r^n\right\}r^{p-m}\;.\;.\;.\;.\;.\;.\;.\;(3.5)$$

and

$$\left|f^{(m)}(z)\right|\le\left\{\delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\sum_{k=p+n}^{\infty}a_k\right\}$$

$$\le\left\{\delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q))}\right\}$$

$$\le\left\{\delta(p,m)+\delta(p+n,m)\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}r^n\right\}r^{p-m}\;\ldots\ldots(3.6)$$

The proof of theorem 2 is done.

Putting m=0 in previous theorem 2 , we get the following corollary

Corollary 2. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\;(p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then

$|z| = r < 1$

$$|f(z)| \geq \left\{1 - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^p \ \ldots\ldots (3.7)$$

and

$$|f(z)| \leq \left\{1 + \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^p \ \ldots\ldots (3.8)$$

The result is sharp.

Putting m = 1 in previous Theorem 2, we get

Corollary 3. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p, q, A, B, \lambda, \alpha, l, \beta)$ then $|z| = r < 1$ we have

$$|f'(z)| \geq \left\{p - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^{p-1} \ \ldots\ldots (3.9)$$

and

$$|f'(z)| \leq \left\{p + \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)} r^n\right\} r^{p-1} \ \ldots\ldots (3.10)$$

The result is sharp

Remark : Taking $q = 0$ *and* $n = 1$ in Corollaries 2 and 3 we obtain the result obtained by Aouf [3, theorem 2]

4.Radius of Starlikeness and Radius of convexity

Theorem 3. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p, q, A, B, \lambda, \alpha, l, \beta)$ then $f(z)$ is $p-$ valent close to convexity of order $\eta\ (0 \leq \eta \leq p)$ in $|z| \leq r_1$ where

$$r_1 = inf\left\{\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)}\right\}^{\frac{1}{k-p}} \quad (k \geq n+p, p, n \in N)\ \ldots\ldots (4.1)$$

The result is sharp , the extremal function given by (2.4).

Proof : we must show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \eta \quad \text{for } |z| \leq r_1 \quad \ldots..(4.2)$$

where r_1 is given by (4.1). Indeed we find from (1.2) that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+n}^{\infty} k a_k |z|^{k-p}$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \eta$$

If

$$\sum_{k=p+n}^{\infty}\left(\frac{k}{p-\eta}\right) a_k |z|^{k-p} \leq 1 \ \ldots\ldots\ldots.(4.3)$$

By theorem 1 , (4.3) will be hold if

$$\left(\frac{k}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q2}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)$$

Then

$$|z| \leq \left(\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)^{\frac{1}{k-p}} \ldots\ldots\ldots\ldots(4.4)$$

The result is follow from (4.4)

Theorem 4. Assume function $f(z)$ is defined by (1.2) be in the class $T_n^*\ (p, q, A, B, \lambda, \alpha, l, \beta)$then $f(z)$ is $p - valent$ starlikeness of order $\eta\ (0 \leq \eta < p)\ in\ |z| \leq r_2$ where

$$r_2 = inf_{k\geq n+p}\left(\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)^{\frac{1}{k-p}} \ldots\ldots\ldots.(4.5)$$

The result is sharp the extremal function given by (2.4)

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \eta \quad \text{for} \quad |z| \leq r_2 \ \ldots\ldots\ldots(4.6)$$

where r_2 given by (4.5). From definition (1.2) that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=n+p}^{\infty} a_k|z|^{k-p}}$$

Thus

$$\left|\frac{zf'(z)}{z^{p-1}} - p\right| \leq p - \eta$$

If

$$\sum_{k=n+p}^{\infty}\left(\frac{k-\eta}{p-\eta}\right) a_k |z|^{k-p} \leq 1\ldots\ldots\ldots(4.7)$$

By using theorem 1 , (4.7) will be true if

$$\left(\frac{k-\eta}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)$$

$$|z| \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right)^{\frac{1}{k-p}} \quad k \geq n+p, n \in N\ \ldots\ldots (4.8)$$

Corollary 4. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*\ (p, q, A, B, \lambda, \alpha, l, \beta)$Then $f(z)$ is in p-valent convex of order η (0 ≤ η < p) in $|z| \leq r_3$, where

$$r_3 = inf_{k\geq n+p}\left\{\frac{[A+B(l-\beta)]p(p-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right\}^{\frac{1}{k-p}}$$

Sharpness is hold, with the extremal given by (2.4).

5.Closure theorems

Theorem 5. Let $\mu_j \geq 0\ for\ j = 1,2, \ldots ., m\ and\ \sum_{j=1}^{m}\mu_j \leq 1$, if function Body Math $f_j(z)$ defined by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k\ (a_{k,j} \geq 0, j = 1,2, \ldots, m) \ \ldots(5.1)$$

are in the class $T_n^*\ (p, q; A, B, \lambda, \alpha, l, \beta)$ for $j = 1,2, \ldots, m$ then the function f(z) defined by

$$f(z) = z^p - \sum_{k=p+n}^{\infty}\left(\sum_{j=1}^{\infty}(\mu_j a_{k,j}) z^k\right) \ldots\ldots (5.2)$$

Is also in the class $T_n^*\ (p, q; A, B, \lambda, \alpha, l, \beta)$

Proof :

Since $f_j(z)$ is in the class $T_n^*\ (p, q, A, B, \lambda, \alpha, l, \beta)$ then by theorem 1 that

$$\sum_{k=p+n}^{\infty}[A + B(l-\beta)]k(k-1)\delta(k-2,q)a_{k,j} \leq \lambda(l-\beta)(1-\alpha)\delta(p-2,q)\ \ldots\ldots.. (5.3)$$

For every j=1,2,… m Hence

$$\sum_{k=p+n}^{\infty}[A + B(l-\beta)]k(k-1)\delta(k-2,q)\left(\sum_{j=1}^{\infty}\mu_j a_{k,j}\right)$$

$$= \sum_{j=1}^{\infty} M_j\left(\sum_{k=p+n}^{\infty}[A + B(l-\beta)]k(k-1)\delta(k-2,q)a_{k,j}\right)$$

$$\leq \sum_{k=p+n}^{\infty}[A + B(l-\beta)]k(k-1)\delta(k-2,q)a_{k,j}\sum_{j=1}^{\infty}\mu_j$$

$$= \lambda(l-\beta)(1-\alpha)\delta(p-2,q)$$

From theorem 1 , it follows that $f(z) \epsilon\ T_n^*\ (p, q; A, B, \lambda, \alpha, l, \beta)$ and so this completes the proof of theorem 5.

Corollary 5. The class $T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ is closed under convq linear combination

Proof :

Let the function $f_j(z)(j=1,2)$ be given by (5.1) be in the class $T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$. It is sufficient to show that the function $f(z)$ defined by

$$f(z)=\mu f_1(z)+(1-\mu)f_2(z)$$

is in the class $T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ .But , taking m=2 , $c_1=\mu$, $c_2=1-\mu$ in theorem 5 , We have the corollary

Theorem 6. Let $f_{p+n-1}(z)=z^p$ and

$$f_k(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k \ ,k\geq p+n$$

…….(6.1)

Then $f(z)$ is in the class $T_n^* \ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=p+n-1}^{\infty} M_k f_k(z) \ldots\ldots\ldots(6.2)$$

Where $\mu_k\geq 0$ and $\sum_{k=p+n-1}^{\infty}\mu_k=1$

Proof :

Assume that

$$f(z)=\sum_{k=p+n-1}^{\infty}\mu_k f_k(z)$$

$$=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\mu_k z^k \ldots\ldots(6.3)$$

Then it follows that

$$\sum_{k=p+n}^{\infty}\left(\frac{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)*\left(\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\mu_k z^k\right)$$

$$\leq\sum_{k=p+n}^{\infty}\mu_k=\left(1-\mu_{p+n-1}\right)\leq 1$$

Hence by theorem 1, we have

$f(z)\in T_n^* \ (p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$.

Conversely , assume that the function $f(z)$defined by (1.2) belongs to the class $T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$, then

$$a_k\leq\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k$$

Setting

$$\mu_k=\frac{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_k$$

Where

$$\mu_{p+n-1}=1-\sum_{k=p+n}^{\infty}\mu_k$$

We can see that $f(z)$ can be expressed in the form (5.5) ,this completes the proof of theorem 6

Corollary 6. The extreme point of the class T_n^* $(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ are the function $f_p(z)=z^p$ and

$$f_k(z)=z^p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k \quad k\geq p+n$$

6.Modified Hadamard products

Assume function $f_j(z)(j=1,2)$ defined by (5.1) the Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z^p-\sum_{k=p+n}^{\infty}a_{k,1}a_{k,2}z^k=$$
$$(f_1*f_2)(z)\ldots.(6.1)$$

Theorem 7. Let the function $f_j(z)(j=1,2)$ defined by (5.1) be in the class $T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ $,then$ $(f_1*f_2)(z)$ be in the class $T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$,where

$$\sigma=1-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)](p+1)(p+n-1)\delta(p+n-2,q)} \quad (n\in N)\ldots..(6.2)$$

The result is sharp for the function $f_j(z)$(j = 1, 2) defined by

$$f_j(z)=z^p+\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{\sum_{k=p+n}^{\infty}[A+B(l-\beta)](p+1)(p+n-1)\delta(p+n-2,q)}z^{p+q}(6.3)$$

Proof : Depending the technique used earlier by Schild and Silverman [7], we must to show the largest σ such that

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1))\delta(k-1,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,1}a_{k,2}\leq 1\ldots\ldots(6.4)$$

We have $f_j(z)\in T_n^* \ (p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)\ (j=1,2)$ then

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,1}\leq 1\ldots\ldots(6.5)$$

and

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,2}\leq 1\ldots\ldots(6.6)$$

By using Cauchy Scharz inequality , we have

$$\sum_{k=p+n}^{\infty}\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\sqrt{a_{k,1}a_{k,2}}\leq 1\ldots\ldots(6.7)$$

It is sufficient to show that

$$\frac{1}{1-\sigma}a_{k,1}a_{k,2}\leq\frac{1}{(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}$$

……..(6.8)

or

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{1-\sigma}{1-\alpha}$$……..(6.9)

Hence in night of the inequality (6.9), it is sufficient to prove that

$$\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\leq\frac{(1-\sigma)}{(1-\alpha)} \quad (k\geq p+n)\ldots\ldots..(6.10)$$

From (6.10) we have

$$\sigma\leq 1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ldots\ldots..(6.11)$$

In the next , we defined the function $R(k)$by

$$R(k)=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ldots\ldots.(6.12)$$

We note that $R(k)$ is an in creasing function of k (k$\geq p+n$), therefore , we caclud that

$$\sigma\leq R(p+n)=$$
$$1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}\ldots\ldots(6.13)$$

The proof is completes

Putting β = 1 in Theorem 7, we obtain the following corollary.

Corollary 7. Let the functions $f_j(z)(j=1,2)$defined by (1.2) be in the class T_n^* (p, q; A , B , λ , α , l) Then where $\gamma=1$

The result is sharp.

Corollary 8. For $f_1(z) and\ f_2(z)$ as in Theorem 7 , the function

$$h(z)=z-\sum_{k=2}^{\infty}\sqrt{a_{k,1}a_{k,2}}\ z^k$$

belongs to the class $T_n^*\ (p,q;A\ ,B\ ,\lambda\ ,\alpha\ ,l)$.

This result follows from the Cauchy-Schwarz inequality (6.7). It is sharp for the same functions as in Theorem 7.

Theorem 8. Let the functions $f_j(z)(j=1,2)$ defined by (5.1) be in the class$T_n^*\ (p,q;A\ ,B\ ,\lambda\ ,\alpha\ ,l)$. Then the function

$$h(z)=z^p-\sum_{k=p+n}^{\infty}\sqrt{\left(a^2{}_{k,1}a^2{}_{k,2}\right)}\ z^k\ \text{…….(6.14)}$$

belongs to the class $T_n^*\ (p,q;A\ ,B\ ,\lambda\ ,\varsigma\ ,l,\beta), where$

The result is sharp for the functions $f_j(z)(j=1,2)$ defined by (6.3)

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=p+n}^{\infty}\left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2 a^2{}_{k,1}\leq$$

$$\sum_{k=p+n}^{\infty}\left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}a^2{}_{k,1}\right]^2\leq 1\ \dots.(6.16)$$

and

$$\sum_{k=p+n}^{\infty}\left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2 a^2{}_{k,2}\leq$$

$$\sum_{k=p+n}^{\infty}\left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}a^2{}_{k,2}\right]^2\leq 1\ \dots.(6.17)$$

It follows from (6.16) and (6.17) that

$$\sum_{k=p+n}^{\infty}\frac{1}{2}\left[\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2\left(a^2{}_{k,1}a^2{}_{k,2}\right)\leq$$

1Therefore, we need to find the largest ζ such that

$$\frac{\lambda(l-\beta)\delta(p-2,q)(1-\zeta)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\leq$$

$$\frac{1}{2}\left[\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\right]^2\ \dots\dots\dots.(6.19)$$

that is, that

$$\zeta\geq 1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ \dots\dots\dots.(6.20)$$

since

and Theorem 8 follows at once.

$$D(k)=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)}\ \dots\dots(6.21)$$

is an increasing function of $k(k\geq p+n)$, we readily have

$\zeta\geq D(p+n)$

$$=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)}\ \dots(6.22)$$

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Local Fusion Graphs of Double Covers of Certain Mathieu Groups and Their Automorphism Groups

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***Abstract*:** Let G a finite group and X a subset of G. The local fusion graph denoted by Ƒ(G,X) has a vertex set X with two distinct element $x \neq y \in X$ are adjacent if the group generated by x and y, $\langle x,y \rangle$, is dihedral group, of order 2n, n odd. In this paper we prove that the local fusion graphs for Mathieu groups and their Automorphism groups has diameter 2.

***Keywords*:** Double covers of Mathieu groups, Local Fusion Graphs, Collapsed Adjacency Matrices, Diameters.

1. Introduction

Recently, the study of the action of the group on graph has been shown to be effective when studying properties of a group. Suppose that G a group with finite order and X class of involution in G, the local fusion graph denoted by Ƒ(G,X) has a vertex set X with two distinct vertices are connected if the group generated by x and y, $\langle x,y \rangle$, is dihedral group of order 2n, n odd, so x conjugate to y in $\langle x,y \rangle$. Studying the structure of groups by using the associated local fusion graphs can be seen in [1]–[3] where X taken to be a conjugacy class of involution. This paper deal with local fusion graphs computationally , the computer algebra systems Magma [4] and GAP [5] have been employed for this purpose. Also, the group representation which define in Magma and GAP can be obtained from the online Atlas of Group Representations [6]. One can show immediately that G induces graph automorphisms on the local fusion graph Ƒ(G,X) (by conjugation) and acts transitively on the graph vertices. For distinct x, $y \in X$, a distance between x and y, d(x,y), is a shortest path between x and y. Also the i^{th} disc of the element $x \in X$, $\Delta_i(x)$, is the set of vertices of Ƒ(G,X) which has distance i from x, also, we may let Diam(Ƒ(G,X)) to be the diameter of Ƒ(G,X) . Let $x \in G$ the Centralizer (the set of elements in G commute with x) in G of x G_x (= $C_G(x)$). Clearly , $\Delta_i(x)$ equal a union of certain $C_G(x)$ -orbits. Finally, we should mentioned that the notations of this groups from Atlas [7]. The main goal of this paper is to investigate the local fusion graphs for Mathieu groups and their Automorphism and we prove computationally both graphs have diameter 2.

2. Main Results

In the 19th century Emile Mathieu discovered the Mathieu groups which are the first family of sporadic simple groups (see [8], [9]).

In this paper we study the local fusion graph for the following groups:

- $2.M_{12}.2$ for the class 2D (class of elements of order 2) with size 1584.
- $2.M_{22}.2$ for the class 2F (class of elements of order 2) with size 2772.

As the rest of the classes divide into different classes with isomorphic local fusion graph see [10]. Let t be a fixed involution (element of order 2) in either 2D or 2F. Since the center of the above groups is cyclic group of order 2, generated by involution say ς, then by [6] one can see that $t\varsigma \in t^G$. Thus for any involution $\eta \in t^G$, the element tς has even order in G. For that reason we assume that X=2D\{tς} or X=2F\{tς}. Magma can provide a code to find the permutation rank of $C_G(t)$ on 2D or 2F which is equal to the number of $C_G(t)$-orbits under the action on 2D or 2F by conjugation, and this for the case 2D and 2F is 27 and 28, respectively.

Let C be a Conjugate class in G so that($C=\{xcx^{-1} | x \in G, c \in C\}=c^G$) , then the set X_C defined to be the set of all element $x \in X$, such that $tx \in X$. Obviously, $C_G(t)$ breaks up into suborbits by its action on X_C, C all over the classes of G. And by [11] the following formula gives us the size of the set X_C

$$|X_C| = \left(\frac{|G|}{|C_{G(g)}||C_{G(t)}|} \right) \left(\sum_{\chi \in Irr} \frac{\chi(g)\chi(t)\chi(t)}{\chi(1)} \right) \quad (1)$$

Where the sum is over all of the irreducible characters χ(g) of G, for $g \in G$. The previous formula is calculated by GAP using the code **"Class Multiplication Coefficient"**. Thus the size of X_C now available computationally.

Now we explain a procedure to find the $\Delta_i(t) \cap X_C$ for the above groups. In order to do that we define the following algorithm which aim to find suborbit representatives. The structure of this algorithm summarized as follows :

Algorithm 1.

Input: G is either 2.M_{12}.2 or 2.M_{22}.2, t involution in 2D or 2F,respectively;
i: r −→ Random(t^G \{t})
ii: set Reps −→ {r} and CR −→ r^{Gt} .
iii: **for** x ∈ t^G\{t} check **if** x ∉ CR (symbol for r^{Gt}), **then**
iv: CR −→ CR ∪ {x^{Gt}}; and Reps −→ Reps ∪ {x}.
Output: The set of suborbit representatives.

The next result cope with the diameters of the local fusion graphs Ƒ(2.M_{12}.2, 2D\{ tς}) and Ƒ(2.M_{22}.2, 2F\{ tς}).

Theorem 1. The Diameter of local fusion graphs Ƒ(2.M_{12}.2, 2D\{ tς}) and Ƒ(2.M_{22}.2, 2F\{ tς}) equal 2

Proof: We have form the output of **Algorithm 1** we find 27 and 28 representatives for C_G(t)-orbits for the graphs Ƒ(2.M_{12}.2, 2D\{ tς}) and Ƒ(2.M_{22}.2, 2F\{ tς}). Furthermore, the Magma code **"Is Conjugate"** is in service to find the set of conjugacy classes such that $X_C \neq \phi$.
From that we can get the G-classes such that X_C is non-empty for both graphs:
{2ABC,3AB,4A,5B,6ABCD,10A,11A,12A,20A,22}
and {2ADE,3A,4CDF,5A,6ABC,10A,11A,22A}, respectively.
The graph Ƒ(2.M_{12}.2, 2D\{ tς}) has 16 class make $X_C \neq \phi$. Obviously, $X_{\{3AB,5A,11A\}}$ in the Δ_1(t) and the reminder classes cannot be in Δ_1(t) this because they have even order if we multiply their representative by t. Now to check the reminder classes lie in Δ_2(t) we first find the whole Δ_1(t) and then search for a random element y∈ Δ_1(t) and we see that there is an element z in X_C such that C is even class with property <y,z> is dihedral group of order 2n, n odd.
Thus:

$$\text{Diam } (Ƒ(2.M_{12}.2, 2D\backslash\{ t\varsigma\})) = 2.$$

Similar approach could be utilized to prove that:

$$\text{Diam } (Ƒ(2.M_{22}.2, 2F\backslash\{ t\varsigma\})) = 2 \qquad \square$$

.
The proof of Theorem 1 computationally can be explained as follows:

1. Use the magma code **"Is Conjugate"** break up the set X_C into the non-empty classes.
2. Δ_1(t) representative is the one in X_C such that C is odd call this set of representative by SubRep. Then
3. $\Delta_1(t)=\bigcup_{x \in SubRep} Conugate(C_G(t), x)$.
4. The reminder class named by RemSubRep
5. For y in -Δ_1(t) there is an element x in RemSubRep such that yx has odd order.
6. $\Delta_2(t)=\bigcup_{x \in RemSubRep} Conugate(C_G(t), x)$.

The structure of the local fusion graphs Ƒ(2.M_{12}.2, 2D\{ tς}) and Ƒ(2.M_{22}.2, 2F\{ tς}) are described in the next result.

Theorem 2 The discs structural of local fusion graphs Ƒ(2.M12.2, 2D\{ tς}) and Ƒ(2.M_{22}.2, 2F\{ tς}) can be explain in the following tables:

Table1 Ƒ(2.M_{12}.2, 2D\{ tς})

X_C G-conjugacy Classes	Δ_1(t)	Δ_2(t)
3A	20,20	
3B	60	
5A	60,60	
11A	120,120	
2BC		15
4A		30,2
6A		20,20
6B		20
6CD		60,60
10A		60,60
12A		120
20A		120,120
22A		120,120

Table 2 Ƒ(2.M_{22}.2, 2F\{ tς})

X_C G-conjugacy Classes	Δ_1(t)	Δ_2(t)
3A	40,40	
5A	160,160	
11A	320,320	
2DE		5,20
4CD		80
4F		40,40,40,40
6A		40,40
6BC		80,80
10A		160,160
22A		320,320

Proof: Theorem 1 shows that the diameters for both graphs are equal 2. Also, the Gap code "**Class Multiplication Coefficien**t" may apply to find the sizes of X_C, which break up to suborbits. To calculate the size of arbitrary suborbits say x ∈X_C we divide the $|C_G(t)|$ by $|C_{CG(t)}(x)|$ which can be done by using the magma code

"Order(Centraliser(G,t)/ Centraliser(Centraliser(G,t),x))". □

3. The Collapsed Adjacency Matrices

For a given two C_G(t)-orbits, say o_i, o_j the collapsed adjacency matrix for the local fusion graph Ƒ(G,X) has entry, represent the number of the edges in the orbit o_j that are connected to a single vertex in the orbit o_i. In the following matrices we change each orbit in **Table 1** and **Table 2** with $\Delta_i^j(t)$ Increasingly, also we let $\Delta_0^1(t)$=t . The next tables 3,4 presents the collapsed adjacency matrix for the local fusion graph Ƒ(G,X), such that **Table 3** gives the details for the graph Ƒ(2.M_{12}.2, 2D\{ tς}), whereas **Table 4** provides the information for the graph Ƒ(2.M_{22}.2, 2F\{ tς}):

Table3: The Collapsed Adjacency Matrices for Ƒ(2.M_{12}.2, 2D\{ tς})

Class	Δ_0^1	Δ_1^1	Δ_1^2	Δ_1^3	Δ_1^4	Δ_1^5	Δ_1^6	Δ_1^7	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_2^9	Δ_2^{10}	Δ_2^{11}	Δ_2^{12}	Δ_2^{13}	Δ_2^{14}	Δ_2^{15}	Δ_2^{16}	Δ_2^{17}	Δ_2^{18}
Δ_0^1	0	20	20	60	60	60	120	120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	20	17	0	18	30	24	60	18	9	0	1	9	10	0	18	24	12	30	6	6	12	36	36	36	18	30
Δ_1^2	20	0	17	18	30	24	18	60	9	0	1	9	0	10	18	12	24	6	30	6	12	36	36	36	30	18
Δ_1^3	60	6	6	20	15	19	30	30	0	6	0	8	6	6	24	14	14	20	20	24	12	36	36	28	40	40
Δ_1^4	60	10	10	15	24	23	42	42	8	6	0	8	2	2	24	18	18	14	14	8	16	32	36	36	26	26
Δ_1^5	60	8	8	19	23	28	36	36	4	2	0	8	4	4	12	16	16	18	18	16	16	36	28	36	34	34
Δ_1^6	120	10	3	15	21	18	40	28	5	4	1	9	5	3	20	17	17	24	18	13	17	34	35	35	28	40
Δ_1^7	120	3	10	15	21	18	28	40	5	4	1	9	3	5	20	17	17	18	24	13	17	34	35	35	40	28
Δ_2^1	15	12	12	0	32	16	40	40	13	0	0	0	0	0	24	12	12	28	28	24	8	32	32	32	32	32
Δ_2^2	15	0	0	24	24	8	32	32	0	13	0	0	12	12	0	28	28	12	12	32	16	32	32	32	40	40
Δ_2^3	30	10	10	0	0	0	60	60	0	0	1	0	10	10	0	0	0	0	0	0	0	60	60	60	60	60
Δ_2^4	2	6	6	16	16	16	36	36	0	0	0	13	6	6	16	20	20	20	20	16	16	56	24	24	36	36
Δ_2^5	20	10	0	18	6	12	30	18	0	9	1	9	17	0	18	30	6	24	12	30	24	36	36	36	18	60
Δ_2^6	20	0	10	18	6	12	18	30	0	9	1	9	0	17	18	6	30	12	24	30	24	36	36	36	60	18
Δ_2^7	20	6	6	24	24	12	40	40	6	0	0	8	6	6	20	20	20	14	14	15	19	36	36	28	30	30
Δ_2^8	0	8	4	14	18	16	34	34	3	7	0	10	10	2	20	19	16	18	12	14	18	32	34	34	36	48
Δ_2^9	0	4	8	14	18	16	34	34	3	7	0	10	2	10	20	16	19	12	18	14	18	32	34	34	48	36
Δ_2^{10}	0	10	2	20	14	18	48	36	7	3	0	10	8	4	14	18	12	19	16	18	16	32	34	34	34	34
Δ_2^{11}	0	2	10	20	14	18	36	48	7	3	0	10	4	8	14	12	18	16	19	18	16	32	34	34	34	34
Δ_2^{12}	0	2	2	24	8	16	26	26	6	8	0	8	10	10	15	14	14	18	18	24	23	32	36	36	42	42
Δ_2^{13}	0	4	4	12	16	16	34	34	2	4	0	8	8	8	19	18	18	16	16	23	28	36	28	36	36	36
Δ_2^{14}	0	6	6	18	16	18	34	34	4	4	1	14	6	6	18	16	16	16	16	16	18	32	39	39	34	34
Δ_2^{15}	0	6	6	18	18	14	35	35	4	4	1	6	6	6	18	17	17	17	17	18	14	39	44	31	35	35
Δ_2^{16}	0	6	6	14	18	18	35	35	4	4	1	6	6	6	14	17	17	17	17	18	18	39	31	44	35	35
Δ_2^{17}	0	3	5	20	13	17	28	40	4	5	1	9	3	10	15	18	24	17	17	21	18	34	35	35	40	28
Δ_2^{18}	0	5	3	20	13	17	40	28	4	5	1	9	10	3	15	24	18	17	17	21	18	34	35	35	28	40

Table 4 : The Collapsed Adjacency Matrices for Ƒ(2.M$_{22}$.2, 2F\{ tς})

Class	Δ_0^1	Δ_1^1	Δ_1^2	Δ_1^3	Δ_1^4	Δ_1^5	Δ_1^6	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_2^9	Δ_2^{10}	Δ_2^{11}	Δ_2^{12}	Δ_2^{13}	Δ_2^{14}	Δ_2^{15}	Δ_2^{16}	Δ_2^{17}	Δ_2^{18}	Δ_2^{19}	Δ_2^{20}
Δ_0^1	**0**	**40**	**40**	**160**	**160**	**320**	**320**	**5**	**20**	**5**	**20**	**80**	**80**	**40**	**40**	**40**	**40**	**40**	**40**	**80**	**80**	**80**	**80**	**160**	**160**	**320**	**320**
Δ_1^1	**40**	**25**	**0**	**56**	**80**	**112**	**136**	**0**	**2**	**12**	**1**	**36**	**32**	**0**	**20**	**16**	**12**	**32**	**0**	**32**	**36**	**32**	**32**	**40**	**72**	**80**	**144**
Δ_1^2	**40**	**0**	**25**	**80**	**56**	**136**	**112**	**0**	**2**	**12**	**1**	**36**	**32**	**20**	**0**	**12**	**16**	**0**	**32**	**36**	**32**	**32**	**32**	**72**	**40**	**144**	**80**
Δ_1^3	**160**	**14**	**20**	**77**	**68**	**130**	**122**	**4**	**3**	**10**	**2**	**28**	**38**	**16**	**12**	**14**	**16**	**10**	**18**	**30**	**30**	**34**	**36**	**52**	**40**	**114**	**102**
Δ_1^4	**160**	**20**	**14**	**68**	**77**	**122**	**130**	**4**	**3**	**10**	**2**	**28**	**38**	**12**	**16**	**16**	**14**	**18**	**10**	**30**	**30**	**36**	**34**	**40**	**52**	**102**	**114**
Δ_1^5	**320**	**14**	**17**	**65**	**61**	**125**	**104**	**6**	**3**	**9**	**2**	**27**	**38**	**19**	**12**	**13**	**16**	**10**	**18**	**29**	**31**	**35**	**36**	**57**	**51**	**126**	**116**
Δ_1^6	**320**	**17**	**14**	**61**	**65**	**104**	**125**	**6**	**3**	**9**	**2**	**27**	**38**	**12**	**19**	**16**	**13**	**18**	**10**	**31**	**29**	**36**	**35**	**51**	**57**	**116**	**126**
Δ_2^1	**5**	**0**	**0**	**32**	**32**	**96**	**96**	**17**	**0**	**0**	**0**	**64**	**0**	**16**	**16**	**16**	**16**	**24**	**24**	**52**	**52**	**20**	**20**	**80**	**80**	**144**	**144**
Δ_2^2	**20**	**16**	**16**	**96**	**96**	**192**	**192**	**0**	**1**	**0**	**0**	**0**	**0**	**16**	**16**	**0**	**0**	**8**	**8**	**0**	**0**	**0**	**0**	**64**	**64**	**128**	**128**
Δ_2^3	**5**	**24**	**24**	**80**	**80**	**144**	**144**	**0**	**0**	**17**	**0**	**0**	**64**	**16**	**16**	**16**	**16**	**0**	**0**	**20**	**20**	**52**	**52**	**32**	**32**	**96**	**96**
Δ_2^4	**20**	**8**	**8**	**64**	**64**	**128**	**128**	**0**	**0**	**0**	**1**	**0**	**0**	**16**	**16**	**0**	**0**	**16**	**16**	**0**	**0**	**0**	**0**	**96**	**96**	**192**	**192**
Δ_2^5	**80**	**18**	**18**	**56**	**56**	**108**	**108**	**16**	**0**	**0**	**0**	**21**	**16**	**16**	**16**	**18**	**18**	**16**	**16**	**18**	**18**	**16**	**16**	**76**	**76**	**152**	**152**
Δ_2^6	**80**	**16**	**16**	**76**	**76**	**152**	**152**	**0**	**0**	**16**	**0**	**16**	**21**	**16**	**16**	**18**	**18**	**18**	**18**	**16**	**16**	**18**	**18**	**56**	**56**	**108**	**108**
Δ_2^7	**40**	**0**	**20**	**64**	**48**	**152**	**96**	**8**	**2**	**8**	**2**	**32**	**32**	**37**	**0**	**12**	**12**	**0**	**20**	**32**	**36**	**36**	**32**	**64**	**48**	**152**	**96**
Δ_2^8	**40**	**20**	**0**	**48**	**64**	**96**	**152**	**8**	**2**	**8**	**2**	**32**	**32**	**0**	**37**	**12**	**12**	**20**	**0**	**36**	**32**	**32**	**36**	**48**	**64**	**96**	**152**
Δ_2^9	**40**	**16**	**12**	**56**	**64**	**104**	**128**	**8**	**0**	**8**	**0**	**36**	**36**	**12**	**12**	**17**	**16**	**12**	**16**	**36**	**32**	**36**	**32**	**64**	**56**	**128**	**104**
Δ_2^{10}	**40**	**12**	**16**	**64**	**56**	**128**	**104**	**8**	**0**	**8**	**0**	**36**	**36**	**12**	**12**	**16**	**17**	**16**	**12**	**32**	**36**	**32**	**36**	**56**	**64**	**104**	**128**
Δ_2^{11}	**40**	**32**	**0**	**40**	**72**	**80**	**144**	**12**	**1**	**0**	**2**	**32**	**36**	**0**	**20**	**12**	**16**	**25**	**0**	**32**	**32**	**36**	**32**	**56**	**80**	**112**	**136**
Δ_2^{12}	**40**	**0**	**32**	**72**	**40**	**144**	**80**	**12**	**1**	**0**	**2**	**32**	**36**	**20**	**0**	**16**	**12**	**0**	**25**	**32**	**32**	**32**	**36**	**80**	**56**	**136**	**112**
Δ_2^{13}	**80**	**16**	**18**	**60**	**60**	**116**	**124**	**13**	**0**	**5**	**0**	**18**	**16**	**16**	**18**	**18**	**16**	**16**	**16**	**21**	**16**	**16**	**18**	**72**	**68**	**144**	**140**
Δ_2^{14}	**80**	**18**	**16**	**60**	**60**	**124**	**116**	**13**	**0**	**5**	**0**	**18**	**16**	**18**	**16**	**16**	**18**	**16**	**16**	**16**	**21**	**18**	**16**	**68**	**72**	**140**	**144**
Δ_2^{15}	**80**	**16**	**16**	**68**	**72**	**140**	**144**	**5**	**0**	**13**	**0**	**16**	**18**	**18**	**16**	**18**	**16**	**18**	**16**	**16**	**18**	**21**	**16**	**60**	**60**	**124**	**116**
Δ_2^{16}	**80**	**16**	**16**	**72**	**68**	**144**	**140**	**5**	**0**	**13**	**0**	**16**	**18**	**16**	**18**	**16**	**18**	**16**	**18**	**18**	**16**	**16**	**21**	**60**	**60**	**116**	**124**
Δ_2^{17}	**160**	**10**	**18**	**52**	**40**	**114**	**102**	**10**	**2**	**4**	**3**	**38**	**28**	**16**	**12**	**16**	**14**	**14**	**20**	**36**	**34**	**30**	**30**	**77**	**68**	**130**	**122**
Δ_2^{18}	**160**	**18**	**10**	**40**	**52**	**102**	**114**	**10**	**2**	**4**	**3**	**38**	**28**	**12**	**16**	**14**	**16**	**20**	**14**	**34**	**36**	**30**	**30**	**68**	**77**	**122**	**130**
Δ_2^{19}	**320**	**10**	**18**	**57**	**51**	**126**	**116**	**9**	**2**	**6**	**3**	**38**	**27**	**19**	**12**	**16**	**13**	**14**	**17**	**36**	**35**	**31**	**29**	**65**	**61**	**125**	**104**
Δ_2^{20}	**320**	**18**	**10**	**51**	**57**	**116**	**126**	**9**	**2**	**6**	**3**	**38**	**27**	**12**	**19**	**13**	**16**	**17**	**14**	**35**	**36**	**29**	**31**	**61**	**65**	**104**	**125**

Conclusion: A good deal of researches have been achieved during this paper. For example, the disc structure of certain local fusion graphs were determined. Moreover, calculating the diameters of these graphs is the most notable of what has been achieved . Also, the collapsed adjacency matrix for the local fusion graphs has been accomplished. Finally, computational approaches were most applied for analyzing the aforementioned local fusion graphs.

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ON SUBCLASS OF MULTIVALENT HARMONIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATION

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Abstract : In this paper , we studied a subclass of multivalent (j-valent) harmonic functions defined by differential operator associated with multiplier transformation , we obtain a coefficients bounds ,distortion bounds and extreme points .
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Keywords : Multivalent functions , harmonic function , differential operator,
multiplier transformation.

1. Introduction:

A function $f = u + iv$ is a continuous and a complex valued harmonic function in a complex domain C , if u and so v are real harmonic in C in simply connected domain $R \subset C$, R is domain we can write $f = h + \overline{g}$, where the functions h and g are analytic functions in R . The function h is called analytic part and the function g is called co- analytic part of the function f . A necessary and sufficient condition for f to be locally univalent and sense – preserving in R is that $|h'(z)| > |g'(z)|$ in R . See [6] .Now , we denoted by $RW(j)$ the class of functions defined by the following form: $f = h + \overline{g}$, that are harmonic multivalent and sense – preserving in the unit disk defined as following $U = \{z \in C : |z| < 1\}$. For f belong to $RW(j)$ we may express the functions h and g as following:

$$h(z) = z^j + \sum_{c=j+1}^{\infty} a_c z^c \quad , \qquad g(z) = \sum_{c=j+1}^{\infty} b_{ck} z^c \quad , |b_c| < 1. \qquad (1)$$

So , for $j \in N, \lambda \geq 0$, the differential operator is defined as following :

$$D_\lambda^{n+j-1} f(z) = D_\lambda^{n+p-1} h(z) + \overline{D_\lambda^{n+j-1} g(z)} \ . \qquad (2)$$

When $j = 1$, D_λ^n denoted of operator introduced by [6]. Also denote $RW^*(j)$ the subclass of $RW(j)$ consisting of all the functions $f = h + \overline{g}$ where h and g defined as :

$$h(z) = z^j - \sum_{c=j+1}^{\infty} |a_c| z^c \quad , \qquad g(z) = -\sum_{c=j+1}^{\infty} |b_c| z^c \quad , |b_c| < 1. \qquad (3)$$

Now, $D_\lambda^{n+j-1} h(z) = z^j + \sum_{c=j+1}^{\infty} [1 + \lambda(c-j)] \varpi(n,c,j) a_c z^c$, (4)

and

$$D_\lambda^{n+j-1} g(z) = \sum_{k=j+1}^{\infty} [1 + \lambda(c-j)] \varpi(n,c,j) b_c z^c \ . \qquad (5)$$

Where $\varpi(n,c,j) = \binom{c+n-1}{n+j-1}$, $n \in N_0$. (6)

Now , the multiplier transformation $I_j(r,\theta)$ defined as following :

$$I_j(r,\hbar) f(z) = I_j(r,\hbar) h(z) + \overline{I_j(r,\hbar) g(z)} \ . \qquad (7)$$

Where

$$I_j(r,\hbar) h(z) = z + \sum_{c=j+1}^{\infty} \Psi(c,j,\hbar)^r a_c z^c , \qquad (8)$$

and

$$I_j(r,\hbar) g(z) = z + \sum_{k=j}^{\infty} \Psi(c,j,\hbar)^r b_c z^c , \qquad (9)$$

where $\Psi(c,j,\hbar)^r = \left(\dfrac{c+\hbar}{j+\hbar}\right)^r \quad , \hbar \geq 0, r \geq 0$. (10)

So , from (2) and (7) , the Hadmard product defined as following :

$$(D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z) = (D_\lambda^{n+j-1} * I_j(r,\hbar)) h(z) + \overline{(D_\lambda^{n+j-1} * I_j(r,\hbar)) g(z)}$$

(11)

where

$$(D_\lambda^{n+j-1} * I_j(r,\hbar)) h(z) = z + \sum_{k=j+1}^{\infty} \gamma(n,c,j,\hbar)^r a_c z^c \ . \qquad (12)$$

And

$$(D_\lambda^{n+j-1} * I_j(r,\hbar)) g(z) = z + \sum_{c=j}^{\infty} \gamma(n,c,j,\hbar)^r b_c z^c , \qquad (13)$$

where

$$\gamma(n,c,j,\hbar)^r = \varpi(n,c,j) * \Psi(c,j,\hbar)^r \ , \qquad (14)$$

Now , we denote by $£_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ the class of all functions defined in (1) such that satisfies the following condition :

$$\operatorname{Re}\left\{ \frac{¥\left((D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{æ\left[\left((D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1) z^{j-2}\right]}{z^{j-2}} \right\} > \Diamond ,$$

(15)

where

$0 < \Diamond < 2\yen, \yen > 0, \lambda \geq 0, \hbar \geq 0, r \geq 0, \ae > 0.$

We note that

$\pounds_{0,0}^{0,0,0}(1,0,1) = S_{\pounds}^{*}, H = \pounds$ studied by Silverman [9],

$\pounds_{\lambda,0}^{0,0,0}(1,0,1) = \pounds(\lambda), H = \pounds$ studied by Yalsin and Öztürk [13],

$\pounds_{0,0}^{0,0,0}(1,\Diamond,1) = N_{\pounds}(\Diamond), \Diamond = \alpha$ class studied by Ahuja and Jahangiry [1],

$\pounds_{\lambda,0}^{n,0,0}(1,0,1) = \pounds_{\lambda}^{n}, H = \pounds$ class studied by authors in [7],

$\pounds_{\lambda,0}^{n,0,0}(j,\Diamond,1) = \pounds_{\lambda}^{n}(j,\Diamond), p = j, \Diamond = \alpha, H = \pounds$ class studied by ALshaqsi and Darus in [11].

Also we see that for the analytic part the class $\pounds_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen), p = j, \theta = \hbar, \tau = \yen, \mu = \ae$ was studied by Goel and Sohi [8].

And so the operator $I_j(r,\hbar)$ was studied by Tehranchin and Kulkarni [12] , Atshan [2], N. E. Cho and T. H. Kim [4], N.E. Cho and Srivastava [5], Saurabh Porwal [10] , J. J. Bhamar and S. M. Khairnar [3].

So ,we denoted by $\eth_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen)$ the subclass of $\pounds_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen)$, where

$$\eth_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen) = RW(j) \cap \pounds_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen). \qquad (16)$$

2.Cofficients Bounds:

In the following theorem , we introduced coefficients bounds of a function in the class $\pounds_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen)$.

Theorem 1:Let $f = h + \overline{g}$, such that the functions h and g are defined in (1) . Let

$$\sum_{c=j}^{\infty} c[1+\lambda(c-j)][\yen + |\ae| j(c-1)]\gamma(n,c,j,\hbar)^{r}(|a_c| + |b_c|) \leq j(2\yen - \Diamond) \qquad (17)$$

Where $a_c = \dfrac{j\yen}{\yen + |\ae| j(j-1)}$,

$0 < \Diamond < 2\yen, \yen > 0, \lambda \geq 0, \hbar \geq 0, r \geq 0, \ae > 0.$

Then f is harmonic multivalent sense preserving in U and f belong to the class $\pounds_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen)$.

Proof: Let

$$A(z) = \frac{\yen\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{\ae\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}}$$

We using the fact $\operatorname{Re}\{A(z)\} \geq \Diamond$ if and only if $|j - \Diamond + A(z)| \geq |j + \Diamond - A(z)|$.

It suffices to show that

$$|j - \Diamond + A(z)| - |j + \Diamond - A(z)| \geq 0. \qquad (18)$$

So ,

$$\left| j - \Diamond + \frac{\yen\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{\ae\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}} \right| -$$

$$- \left| j + \Diamond - \frac{\yen\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} - \frac{\ae\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}} \right|$$

$$= \left| j + \yen - \Diamond + \sum_{c=j+1}^{\infty} \frac{c\yen}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} a_c z^{c-j} + \sum_{k=j}^{\infty} \frac{c\yen}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} \overline{b_c z^{c-j}} \right.$$

$$\left. + \sum_{c=j+1}^{\infty} c\ae[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} a_c z^{c-j} + \sum_{c=j}^{\infty} c\ae[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} \overline{b_c z^{c-j}} \right|$$

$$- \left| j - \yen + \Diamond - \sum_{c=j+1}^{\infty} \frac{c\yen}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} a_c z^{c-j} - \sum_{k=j}^{\infty} \frac{c\yen}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} \overline{b_c z^{c-j}} \right.$$

$$\left. - \sum_{c=j+1}^{\infty} c\ae[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} a_c z^{c-j} - \sum_{c=j}^{\infty} c\ae[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} \overline{b_c z^{c-j}} \right|$$

$$\geq 2\left\{ \yen - \Diamond - \sum_{c=j+1}^{\infty} \frac{c\yen}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} |a_c||z|^{c-j} - \sum_{c=j}^{\infty} \frac{c\yen}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} |b_c||\overline{z}|^{c-j} \right.$$

$$\left. - \sum_{c=j+1}^{\infty} |\ae| c(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} |a_c||z|^{c-j} - \sum_{c=j}^{\infty} |\ae| c(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^{r} |b_c||\overline{z}|^{c-j} \right\}$$

$$= 2\left\{ j(\yen - \Diamond) - \sum_{c=j+1}^{\infty} c[1+\lambda(c-j)][\yen + |\ae| j(c-1)]\gamma(n,c,j,\hbar)^{r} |a_c| - \sum_{c=j}^{\infty} c[1+\lambda(c-j)][\yen + |\ae| j(c-1)]\gamma(n,c,j,\hbar)^{r} |b_c| \right. \geq 0.$$

So, the harmonic mappings

$$f(z) = z^{j} + \sum_{c=j+1}^{\infty} \frac{x_c}{c[1+\lambda(c-j)][\yen + |\ae| j(c-1)]\gamma(n,c,j,\hbar)^{r}} z^{c} + \sum_{c=j}^{\infty} \frac{\overline{y}_c}{c[1+\lambda(c-j)][\yen + |\ae| j(c-1)]\gamma(n,c,j,\hbar)^{r}} (\overline{z})^{c}. \qquad (19)$$

Where ,

$\sum_{c=j+1}^{\infty} |x_c| + \sum_{c=j}^{\infty} |y_c| = j(\yen - \Diamond)$, show that the coefficient bound given by (17) is sharp.

The function of the form (19) are in $\pounds_{0,\hbar}^{n,r,\ae}(j,\Diamond,\yen)$, because

$$\sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r(|a_c|+|b_c|)$$
$$= p\tau + \sum_{c=j+1}^{\infty} |x_c| + \sum_{c=j}^{\infty} |y_c| = j(2¥-\Diamond).$$

In the next theorem , we show that the condition (17) is also a necessary for functions in the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Theorem 2:Let $f = h+\overline{g}$ where the functions h and g are given by (4). Then a function f belong to the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ if and only if

$$\sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r(|a_c|+|b_c|)\le j(2¥-\Diamond). \quad (20)$$

Where $a_c = \dfrac{j¥}{¥+|æ| j(j-1)}$,

$0<\Diamond<2¥, ¥>0, \lambda\ge 0, \hbar \ge 0, r\ge 0, æ>0.$

Proof : The " if " part follows from theorem 1 , upon noting $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥) \subset £_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$. For the " only if " part , assume that f belong to the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$, then by (15) , we get

$$\operatorname{Re}\left\{\frac{¥\left((D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{æ\left[\left((D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}}\right\} > \Diamond$$

$$\operatorname{Re}\left\{\tau - \sum_{c=j+1}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r|a_c|z^{c-j} - \sum_{c=j}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r|b_c|\overline{z}^{c-j}\right.$$
$$\left. - \sum_{c=j+1}^{\infty} æc(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r|a_c|z^{c-j} - \sum_{c=j}^{\infty} æc(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r|b_c|\overline{z}^{c-j}\right\} > \alpha$$

We choosing z to be real and so $æ=|æ|$ and letting $z \to 1^-$, we get required result .

In the following theorem , we obtain distortion bounds for the functions in the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Corollary : If $f \in ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$. Then

$$\sum_{c=j}^{\infty} (|a_c|+|b_c|) \le \frac{j(2¥-\Diamond)}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} \quad (21)$$

Theorem 3: Let f belong the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ and $|z| = r > 1$, then

$$|f(z)| \le (1+a_j)r_1^j + r_1^j \frac{(2¥-\hbar)}{[¥+|æ| j(j-1)]}$$

And

$$|f(z)| \ge (1+a_j)r_1^j - r_1^j \frac{(2¥-\hbar)}{[¥+|æ| j(j-1)]} .$$

Proof:

Let $f \in ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$, so we have

$$|f(z)| \le (1+a_j)r_1^j + r_1^j \sum_{c=j}^{\infty} (|a_c|+|b_c|)$$

Then ,

$$|f(z)| \le (1+a_j)r_1^j + r_1^j \frac{(2¥-\hbar)}{[¥+|æ| j(j-1)]}$$

And so , by similarity we have

$$|f(z)| \ge (1+a_j)r_1^j - r_1^j \frac{(2¥-\hbar)}{[¥+|æ| j(j-1)]}$$

3. Extreme points:

In this section , we shall obtain extreme points for the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Theorem 4: $f \in ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ if and only if f can be expressed by

$$f(z) = \sum_{c=j}^{\infty} (S_c h_c + B_c g_c), \quad (22)$$

where

$$h_j(z) = z^j, \; h_j(z) = z^j - \frac{j(¥-\Diamond)}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} z^c.$$
$(c = j+1, j+2, \ldots)$

and

$$g_k(z) = z^p - \frac{j(¥-\Diamond)}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} (\overline{z})^c.$$
$(c = j+1, j+2, \ldots)$

.

And

$f(z) = \sum_{c=j}^{\infty} (S_c + B_c) = 1$, $S_c \ge 0$, and $B_c \ge 0$, $(c = j+1, j+2, \ldots)$.

In particular , the extreme points of $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ are $\{h_c\}$ and $\{g_c\}$.

Proof :

We can write $f(z)$ as following

$$f(z) = \sum_{k=p}^{\infty} (S_c h_c + B_c g_c) = \sum_{c=j+1}^{\infty} (S_c + B_c) z^j - \frac{j(¥-\Diamond)S_c}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} z^c - \sum_{k=p}^{\infty} \frac{j(¥-\Diamond)B_c}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} (\overline{z})^c$$

$$= z^{p} - \sum_{c=j+1}^{\infty} \frac{j(¥ - \Diamond)S_c}{c[1+\lambda(c-j)][¥ + |æ|\, j(c-1)]\gamma(n,c,j,\hbar)^{r}} z^{c}$$

$$-\sum_{c=j}^{\infty} \frac{j(¥ - \Diamond)B_c}{c[1+\lambda(c-j)][¥ + |æ|\, j(c-1)]\gamma(n,c,j,\hbar)^{r}} (\bar{z})^{c}$$

$$= z^{c} - \sum_{c=j+1}^{\infty} A_c z^{c} - \sum_{c=j}^{\infty} C_c (\bar{z})^{c}.$$

Then from theorem 1 , we have

$$\sum_{c=j+1}^{\infty} c[1+\lambda(c-j)][¥ + |æ|\, j(c-1)]\gamma(n,c,j,\hbar)^{r} A_c -$$

$$-\sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥ + |æ|\, j(c-1)]\gamma(n,c,j,\hbar)^{r} C_c$$

$$= j(¥ - \Diamond)\left(\sum_{c=j}^{\infty} (S_c + B_c) - S_c\right)$$

$$= j(¥ - \Diamond)(1 - S_c) \le j(¥ - \Diamond).$$

Then $f \in T_{\lambda,\theta}^{n,r,\mu}(p,\alpha,\tau)$.

Conversely , let f belong to the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Put

$$S_c = \frac{c[1+\lambda(c-j)][¥ + |æ|\, j(c-1)]\gamma(n,c,j,\hbar)^{r}}{j(¥ - \Diamond)} |a_c|,$$

$(c = j+1, j+2, \ldots)$

And

$$C_c = \frac{c[1+\lambda(c-j)][¥ + |æ|\, j(c-1)]\gamma(n,c,j,\hbar)^{r}}{j(¥ - \Diamond)} |b_c|$$

, $(c = j+1, j+2, \ldots)$.

We obtain

$f(z) = \sum_{c=j}^{\infty} (S_c h_c + C_c g_c)$ as required .

So the proof is complete.

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Global Stability of Cholera Epidemic with General Recovery Rate Involving External Source of Disease

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Abstract: In this article a mathematical model that describes the spread of infectious disease in a population is proposed and studied. This model describes the spread of cholera disease with external source of disease and nonlinear recovery function h(I), The local and global stability of the model is studied. Our results suggest that the basic reproduction number itself is not enough to describe whether cholera will prevail or not. Finally, the global dynamics of this model is studied numerically.

Keyword: Cholera disease, global stability, external source, recovery function.

1. Introduction:

Cholera is a dangerous disease caused by the bacterial Vibrio Cholera. It infects the small intestine. There are many types (strains) of the V. C. Bacterial. Some of them cause more serious illnesses than others. Because of this, some human who get cholera have no symptoms; others have symptoms that are not very bad, and others have very bad symptoms [1-4].

Cholera is a very old epidemic. It still affects many human throughout the world. Estimates from 2010 say that between 3 million and 5 million people get Cholera every year, and 58000-130000 people die from the disease every year. Today, Cholera is called a pandemic. However, it is most common in developing countries, especially in children [5-8]. Cholera is an acute intestinal infection caused by the bacterial V. C. Its dynamics are complicated by the multiple interactions between the human host, the pathogen and the environment which contribute to both direct human-to-human and indirect environment to-human transmission pathways [9].

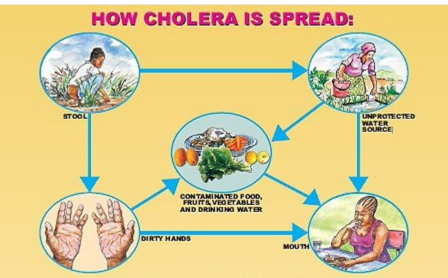


Figure 1: Simplified life cycle of cholera disease

Below, we briefly review some representative mathematical models proposed by various authors. In 2001, [10] extended the model of Capasso and Paveri-Fontana. He added an equation for the dynamics of the susceptible population. And he studied the role of the aquatic reservoir in the endemic and epidemic dynamics of Cholera.

In [11], Pascual et al., Generalized Codeco model by including a 4th equation for the volume of water in which the formative live following [10]. In 2009, Richard I. Joh et al., considered the dynamic of infectious disease for which the primary mode of transmission is indirect and mediated by contact with a contaminated reservoir [12]. Also, Ali and Zhou studied the model for the Cholera disease [13]. In this article is organized as follows. In Section 2, we introduce the generalized model and state the necessary assumptions. In Sections 3, we find the each equilibrium point in this model with derive the B. R. N. using the next-generation matrix approach. In Section 4 and 5, we show the local and global stability of the all equilibrium points. Finally, in order to confirm our obtained results and specify the effects of model's parameters on the dynamical behavior, numerical simulation of the cholera model is performed in Section 6.

2. The mathematical model:

In this articale, we suppose the epidemic model descript the cholera disease by the following equations:

$$\begin{aligned}
\dot{S} &= A - \frac{\beta_{\circ} S(t)B(t)}{K + B(t)} - \beta_1 S(t)I(t) - \beta_2 S(t) - \mu_1 S(t) \\
\dot{I} &= \frac{\beta_{\circ} S(t)B(t)}{K + B(t)} + \beta_1 S(t)I(t) + \beta_2 S(t) - (r + \mu_1 + \delta)I(t) - Ih(I) \qquad (1) \\
\dot{B} &= \eta I(t) - \mu_1 B(t)
\end{aligned}$$

For all time *t*, the population are divided into three classes: a susceptible class *S(t)*, an infectious class *I(t)* and the virus class *B(t)*, that is to say *N=S(t)+I(t)*. All the parameters are positive constant, with descript in the following table:

Table 1: Description of parameters and frequently used symbols.

Parameters	description
A	The birth rate
$\beta_\circ$, β_1	The infection rate
β_2	The infection rate by external source $\beta_2 > 0$
μ_1, μ_2	Dead rate
r	The treatment rate
δ	The disease related death
$h(I) = m/(v + wI)$	The recovery function, with *m,v and w>0*
η	The new infected members from *I* class
K	The carrying capacity

Clearly, the equations of system (1) are continuously differentiable. In fact there is Lipschizan function on R_+^2. Therefore, the solution of system (1) with non-negative initial conditions is uniformly bounded as shown in the following theorem.

Theorem 1: Each the solutions of system (1), which are initiate in R_+^2, are uniformly bounded.

proof: Let *(S(t),I(t))*, be any solutions of the system (1) with non-negative initial conditions *(S(0),I(0))*. Since *N=S(t)+I(t)*, then $\dot{N} = \dot{S} + \dot{I}$, This gives: $\dot{N} + \mu_1 N \le A$.

Now, by using Gronwall lemma [1], it obtains that:

$$N(t) \le \frac{A}{\mu_1}(1 - e^{-\mu_1 t}) + N(0)e^{-\mu_1 t}$$

Therefore, $N(t) \le A/\mu_1$, *as* $t \to \infty$, hence all the solutions of system (1) that in R_+^2, are confined in reign: $\Gamma_H = \{(S, I) \in R_+^2 : N \le A/\mu_1\}$. And the feasible region of pathogen population for system (1) is $\Gamma_Z = \{B : 0 \le B \le \eta A/\mu_1\mu_2\}$. Define $\Gamma = \Gamma_H \times \Gamma_Z$. Let $Int.\Gamma$, denote the interior of Γ. It is easy to verify that the region Γ is positively invariant region with respect to System (1), hence, system (1) will be considered mathematically and epidemiologically well posed in Γ.

4. Local and Global Stable Analysis of $E_\circ$

In this part, the stable analysis of D. F. point $E_\circ(A/\mu_1, 0,0)$ of the system (1) as shown in the following theorems.

Let $Int.\Gamma$, denote the interior of Γ. It is easy to verify that t

3. Existence of Equilibrium Point

In system (1), there are always two biologically feasible points, namely the infection-free equilibrium point $E_\circ(S_\circ, 0,0) = (A/\mu_1, 0,0)$. This point exists when the basic reproduction number $R_\circ < 1$, where:

$$R_\circ = \frac{\beta_\circ \beta_1 \eta A}{K\mu_1\mu_2(r + \mu_1 + \delta + h(0))} \qquad (2)$$

The positive equilibrium point $E_1(S_1, I_1, B_1)$ exists when where:

$$S_1 = \frac{A(K\mu_2 + \eta I_1)}{\eta\beta_\circ I + (K\mu_2 + \eta I_1)[\beta_1 + \beta_2 + \mu_1]} \qquad (3)$$

$$B_1 = \frac{\eta I_1}{\mu_2} \qquad (4)$$

And I_1 is the positive solution of the following equation:

$$D_1 I_1^4 + D_2 I_1^3 + D_3 I_1^2 + D_4 I_1 + D_5 = 0 \qquad (5)$$

Here:

$$D_1 = \beta_1 A \eta^2 w > 0$$

$$D_2 = \eta\big[A(\beta_\circ\eta w + \mu_2\beta_1 K w + \beta_1\eta v + \beta_1 K\mu_2 w + \beta_2\eta w) - \eta(m(\beta_\circ + \beta_1 + \beta_2 + \mu_1) + w(r + \mu_1 + \delta)(\beta_\circ + \beta_1 + \beta_2 + \mu_1)\big]$$

$$D_3 = A\big[\beta_1\mu_2^2 K^2 w + \eta(\beta_\circ\eta v + \beta_1\mu_2 v + \beta_1\mu_2 vK + 2\beta_2\mu_2 wK + \beta_2)\big] + (\beta_1 + \beta_2 + \mu_1)[\eta v + \mu_2 K w + 2\mu_2 K m]\eta\big[\beta_\circ\mu_1 K m + (r + \mu_1 + \delta) \times (\beta_\circ\eta v + \beta_\circ\mu_2 wK + \mu_2 wK(\beta_1 + \beta_2 + \mu_1)\big]$$

$$D_4 = \mu_2 K\big[A(\beta_\circ\eta + \mu_2\beta_1 v + 2\beta_2 v\eta + \beta_2\mu_2 K w) - (r + \mu_1 + \delta)(\beta_\circ\eta v + (2\eta v + \mu_2 K w - \mu_2 K w)(\beta_1 + \beta_2 + \mu_1)\big]$$

$$D_5 = \mu_2^2 K^2 v\big[A\beta_2 - (r + \mu_1 + \delta)(\beta_1 + \beta_2 + \mu_1)\big]$$

Clearly, equation (5) has unique positive root by I_1 if and only if $D_i < 0$, $i = 2,3,4,5$.

Theorem 2: The disease-free equilibrium point of $E_\circ(A/\mu_1, 0,0)$ the system (1) is local asymptotically stable provided that:

$$\beta_1 S_\circ < r + \mu_1 + \delta + h(0) \qquad (6)$$

$$\eta\beta_\circ S_\circ < \min.\{K(\beta_2+\mu_1)^2,\ K\beta_1 S_\circ(\beta_2+\mu_1)(\beta_1 S_\circ - (r+\mu_1+\delta+h(0)),\ K\mu_2(\beta_1 S_\circ-(r+\mu_1+\delta+h(0))\} \quad (7)$$

Proof: The Jacobian matrix of system (1) at $E_\circ$ that denoted by $J(E_\circ)$ and we can be written as:

$$J(E_\circ)=\begin{pmatrix} -(\beta_2+\mu_1) & -\beta_1 S_\circ & -\beta_\circ S_\circ / K \\ \beta_2 & \beta_1 S_\circ-(r+\mu_1+\delta+h(0)) & \beta_\circ S_\circ / K \\ 0 & \eta & -\mu_2 \end{pmatrix}$$

Clearly, the characteristic equation of the Jacobian matrix $J(E_\circ)$ of the system (1) at the disease-free equilibrium point $E_\circ$ is given by $\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0$

Here:

$$A_1=-[a_{11}+a_{22}+a_{33}]$$

$$A_2=[a_{11}a_{22}-a_{12}a_{21}+a_{22}a_{33}-a_{23}a_{32}+a_{11}a_{33}]$$

$$A_3=-[a_{33}a_{12}a_{21}+a_{11}a_{23}a_{32}-a_{11}a_{22}a_{33}-a_{21}a_{32}a_{13}]$$

Further:

$$\Delta=A_1A_2-A_3$$

Now according to (Routh-Hurwitz) criterion $E_\circ(A/\mu_1,0,0)$ will be local stable provided that $A_i>0, i=1,3$ and $\Delta=A_1A_2-A_3>0$ .Clearly: $A_i>0, i=1,3$ with $\Delta=A_1A_2-A_3$ provided that condition (6-7) holds. Hence the proof is complete.

Theorem 3: Let the disease-free equilibrium point $E_\circ$ of System (1) is local stable. Then the basin of attraction of $E_\circ$, say $B(E_\circ)\subset R_+^3$, it is global stable provided the condition is satisfied:

$$\left(\beta_\circ B/_{K+B}+\beta_1 I+\beta_2\right)S_\circ+\eta I<\left(\mu/_S(S-S)^2+(r+\mu_1+\delta+h(0))I+\mu_2 B\right) \quad (8)$$

Proof: Consider the following positive definite function:

$$V_1=\left(S-S_\circ-S_\circ\ln\frac{S}{S_\circ}\right)+I+B$$

Clearly, $V_1: R_+^3\to R$ is a continuously differentiable function such that $V_1(S_\circ,0,0)=0$, and $V_1(S,I,B)>0, \forall(S,I,B)\neq(S_\circ,0,0)$.

Further we have:

$$\dot V_1=\left(\frac{S-S_\circ}{S}\right)\dot S+\dot I+\dot B$$

By simplifying this equation we get:

$$\dot V_1=-\frac{\mu}{S}(S-S_\circ)^2-(r+\mu_1+\delta+h(0))-\mu_2 B+\left(\frac{\beta_\circ B}{K+B}+\beta_1 I+\beta_2\right)S_\circ+\eta I$$

Obviously, $\dot V_1<0$ for each initial point and then V_1 is a Lyap. function provided that condition (8) hold. Thus $E_\circ$ is global stable in $B(E_\circ)$, and that complete the proof.

5. Local with Global Stability Analysis of Positive Point E_1

In this part, the local and global dynamics of system (1) is studied by use the Ruth-Hurwitz and Lyap. function as shown in the theorems.

Theorem 4: The positive point E_1 of the system (1) is local stable if:

$$\beta_1 S_1<r+\mu_1+\delta+h(I_1)+\frac{dh(I_1)I_1}{dI_1} \quad (9)$$

$$\eta\beta_\circ S_1 K<\min.\{(K+B_1)^2L^2,\ \beta_1 S_1(K+B_1)^2 L\times(\beta_1 S_1-(r+\mu_1+\delta+h(I_1)+\frac{dh(I_1)I_1}{dI_1})),\ \mu_2(K+B_1)^2(\beta_1 S_1-(r+\mu_1+\delta+h(I_1)+\frac{dh(I_1)I_1}{dI_1})\} \quad \ldots\ldots\ldots\ldots. (10)$$

Where: $L=\dfrac{\beta_\circ B_1}{K+B_1}+\beta_1 I_1+\beta_2+\mu_1$

Proof: The Jacobian matrix of system (1) at E_1 that denoted by $J(E_1)$ and can be written as:

$$J(E_\circ)=\begin{pmatrix} -(\beta_\circ B_1/_{K+B_1}+\beta_1 I_1+\beta_2+\mu_1) & -\beta_1 S_\circ & -\beta_\circ S_1 K/_{(K+B_1)^2} \\ (\beta_\circ B_1/_{K+B_1}+\beta_1 I_1+\beta_2) & \beta_1 S_1-(r+\mu_1+\delta+h(I_1)+dh(I_1)I_1/_{dI_1}) & \beta_\circ S_1 K/_{(K+B_1)^2} \\ 0 & \eta & -\mu_2 \end{pmatrix}$$

Clearly, the characteristic equation of the Jacobian matrix $J(E_1)$ of the system (1) at the positive point E_1 is given by $\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0$

Here:

$$B_1 = -[b_{11} + b_{22} + b_{33}]$$
$$B_2 = [b_{11}b_{22} - b_{12}b_{21} + b_{22}b_{33} - b_{23}b_{32} + b_{11}b_{33}]$$

$$B_3 = -[b_{33}b_{12}b_{21} + b_{11}b_{23}b_{32} - b_{11}b_{22}b_{33} - b_{21}b_{32}b_{13}]$$

Further:

$$\Delta = B_1B_2 - B_3$$

Now according to (Routh-Hurwitz) criterion E_1 will be local stable provided that $B_1 > 0$; $B_3 > 0$ and $\Delta = B_1B_2 - B_3 > 0$. Clearly, $B_i > 0,\ i = 1,3$ and $\Delta = B_1B_2 - B_3 > 0$, provided that condition (9-10) holds. Hence the proof is complete.

Theorem 5: If the positive point E_1 of System (1) is local stable. Then it is global stable if satisfy the following conditions:

$$\beta_1 S_1 < r + \mu_1 + \delta + X \quad (11)$$

$$\left(\frac{\beta_\circ B_1}{K + B_1} + \beta_1 I_1 + \beta_2 - \beta_1 S_1 \right)^2 < L(r + \mu_1 + \delta + X - \beta_1 S_1) \quad \text{……….. (12)}$$

$$\left(\frac{\beta_\circ SK}{(K+B)(K+B_1)} \right)^2 < \mu_2 L \quad (13)$$

$$\left(\frac{\beta_\circ SK}{(K+B)(K+B_1)} + \eta \right)^2 < \mu_2 (r + \mu_1 + \delta + L - \beta_1 S_1) \quad \text{……….. (14)}$$

Where: $X = \dfrac{mv}{(v + mI)(v + wI_1)}$

Proof: Consider the following positive definite function:

$$V_2 = \frac{(S - S_1)^2}{2} + \frac{(I - I_1)^2}{2} + \frac{(B - B_1)^2}{2}$$

Clearly, $V_2 : R_+^3 \to R$ is a continuously differentiable function such that $V_2(S_1, I_1, B_1) = 0$ and $V_2(S, I, B) > 0,\ \forall (S_1, I_1, B_1) \neq (S, I, B)$. Further, we have:

$$\dot{V}_2 = (S - S_1)\dot{S} + (I - I_1)\dot{I} + (B - B_1)\dot{B}$$

By simplifying this equation we get:

$$\dot{V}_2 = \frac{q_{11}}{2}(S - S_1)^2 - q_{12}(S - S_1)(I - I_1) - \frac{q_{22}}{2}(I - I_1)^2$$
$$- \frac{q_{11}}{2}(S - S_1)^2 + q_{13}(S - S_1)(B - B_1) - \frac{q_{33}}{2}(B - B_1)^2$$
$$- \frac{q_{22}}{2}(I - I_1)^2 + q_{23}(I - I_1)(B - B_1) - \frac{q_{33}}{2}(B - B_1)^2$$

With:

$$q_{11} = L\ ;\ q_{12} = \frac{\beta_\circ B_1}{K + B_1} + \beta_1 I_1 + \beta_2 - \beta_1 S_1$$
$$q_{22} = r + \mu_1 + \delta + X - \beta_1 S_1\ ;\ q_{13} = \beta_\circ SK(K + B)(K + B_1)$$
$$q_{33} = \mu_2\ ;\ q_{23} = \frac{\beta_\circ SK}{(K+B)(K+B_1)} + \eta$$

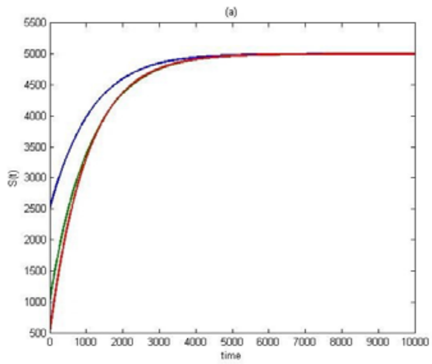
Therefore, according to the conditions (11-14) we obtain that:

$$\dot{V}_2 \le -\left[\sqrt{\frac{q_{11}}{2}}(S - S_1) - \sqrt{\frac{q_{22}}{2}}(I - I_1) \right]^2$$
$$-\left[\sqrt{\frac{q_{11}}{2}}(S - S_1) - \sqrt{\frac{q_{33}}{2}}(B - B_1) \right]^2$$
$$-\left[\sqrt{\frac{q_{22}}{2}}(I - I_1) + \sqrt{\frac{q_{33}}{2}}(B - B_1) \right]^2$$

Clearly, $\dot{V}_2 < 0$ and then V_2 is a Lyap. function provided that the given conditions (11-14) hold. Therefore, E_1 is global stable.

6. Numerical Simulation of system (1)

In this part, the dynamical behavior of system (1) is studied numerically. The objectives of this study are confirming our obtained analytical results and understand the effects of some parameters on the dynamics of system (1). Consequently, system (1) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of all equilibrium points $(E_i, i = \circ,1)$ system (1) has a globally asymptotically stable disease-free equilibrium point as shown in following figures.



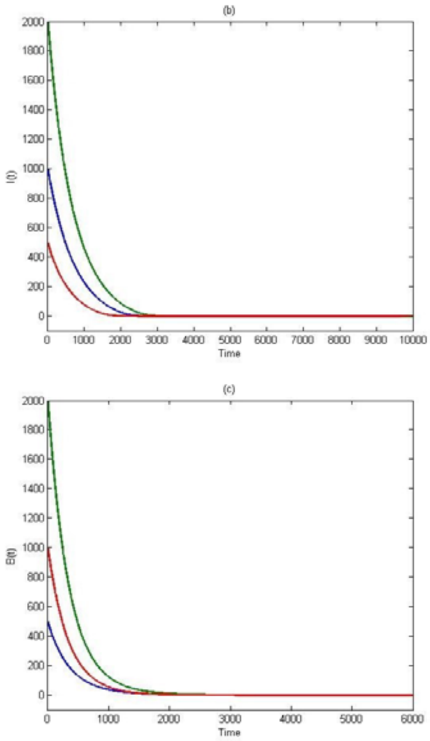
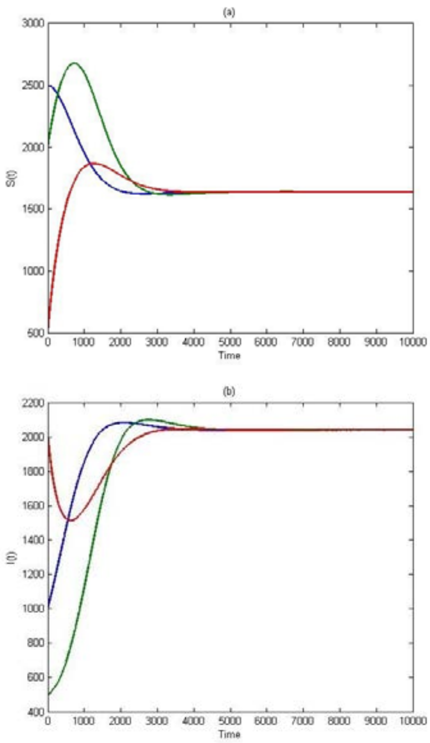


Figure 2: The unique point of system (1) is global stable. In this case, A=*500*, $\beta_\circ$ =0.00001, β_1=0.00001, β_2 =0, η =0.01, μ_1=0.1, μ_2 =0.3, r=0.05, m=1, v=2, w=0.1, K=5, δ =0.01. And the trajectories of system (1) approaches to $E_\circ$ =(5000,0,0), from three initial conditions are (2500,1000,500), (1000,2000,2000) and (500,500,1000).



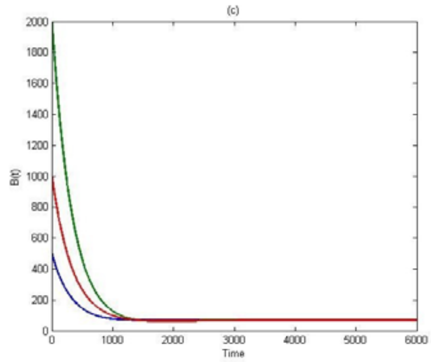
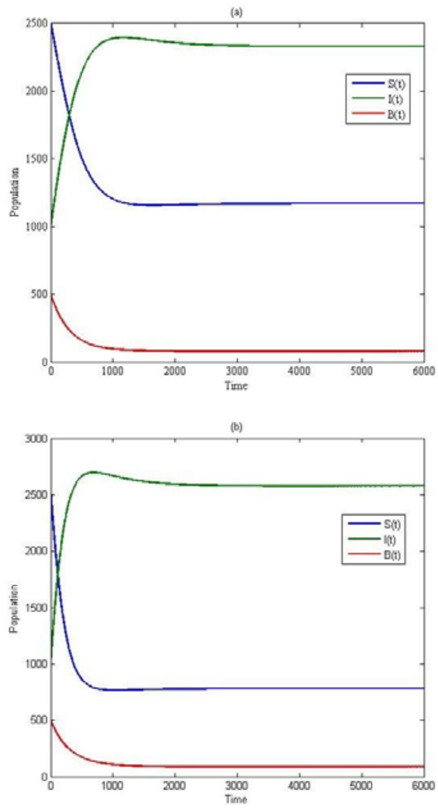


Figure 3: The positive point of system (1) is global stable. In this case, A=*500*, $\beta_\circ$ =0.001, β_1=0.0001, β_2 =0.0001, η =0.01, μ_1=0.1, μ_2 =0.3, r=0.05, m=1, v=2, w=0.1, K=5, δ =0.01. And the trajectories of system (1) approaches to E_1=(1600,2075,95), from three initial conditions are (2500,1000,500), (2000,500,2000) and (500,2000,1000).

Now, we choose the set of hypothetical parameters A=*500*, β_1=0.0001, β_2 =0.0001, η =0.01, μ_1=0.1, μ_2 =0.3, r=0.05, m=1, v=2, w=0.1, K=5, δ =0.01. but we change the infection rate value ($\beta_\circ$ =0.1,0.3,0.5) respectively, we get the trajectories of system (1) still approaches to positive point but the number of *S(t)* decrease while the numbers of the *I(t)* and virus class increases.



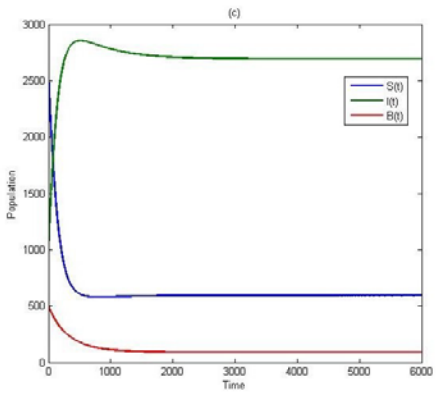
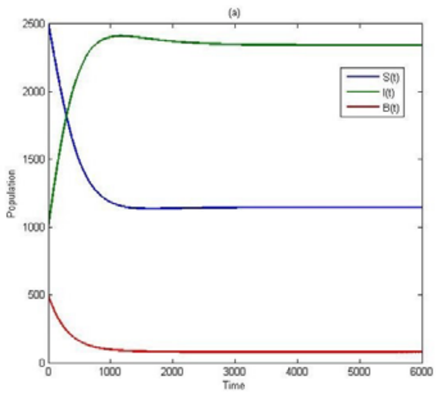
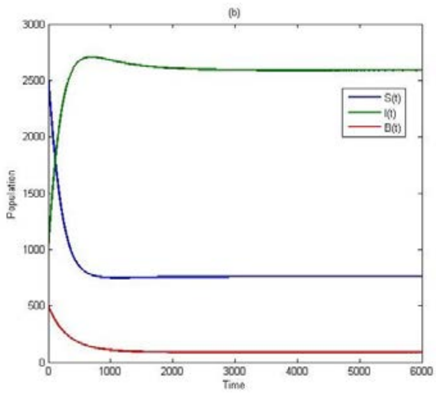


Figure 4: The trajectories of system (1): (a) $\beta_\circ$ =0.1, (b) $\beta_\circ$ =0.3, (c) $\beta_\circ$ =0.5.

Now the effect of external sources in the environment on the dynamics of system (1) is studied by solving the system numerically for the parameters values β_2 =0.1, 0.3, 0.5 respectively, in following figure:





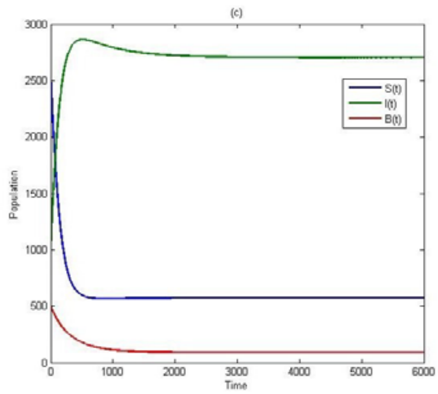
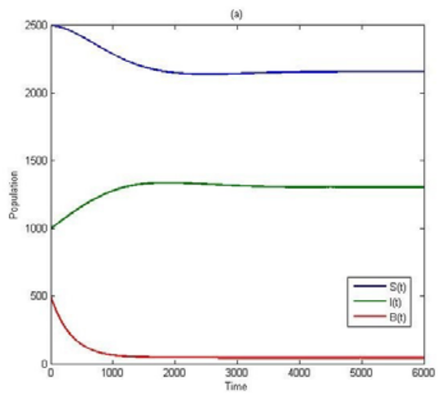
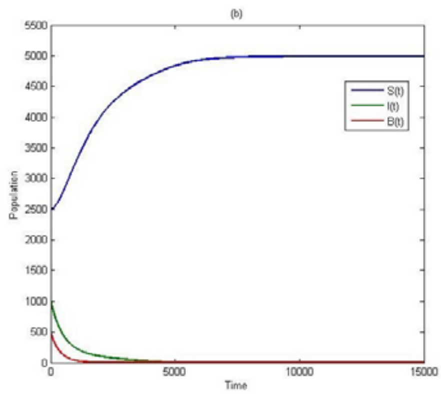


Figure 5: The trajectories of system (1), we use, *A=500*, $\beta_\circ$ =0.001, β_1 =0.0001, η =0.01, μ_1 =0.1, μ_2 =0.3, *r*=0.05, *m*=1, *v*=2, *w*=0.1, *K*=5, δ =0.01, with (a) β_2 =0.1, (b) β_2 =0.3, (c) β_2 =0.5.

According to Figure (5), as the spread of disease by increases the external sources parameter, the trajectory of system (1) approaches to the positive point. In fact as β_2 increases it is observed that the number of *S(t)* individuals decrease and the number of *I(t)* and virus individuals increases.

Clearly, we present the effect of treatment rate that is by change value for *r*=0.1, 0.3, 0.5 respectively, we get the trajectories of system (1) still approaches to positive point but the number of *I(t)* and virus individuals decreases while the *S(t)* individuals is increases.





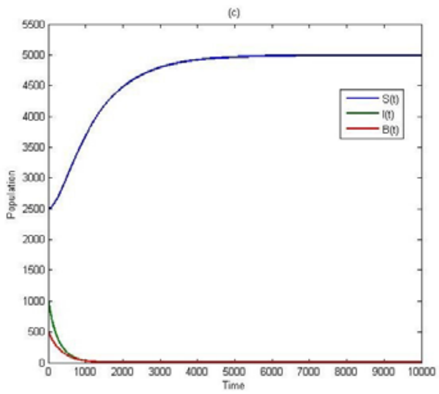
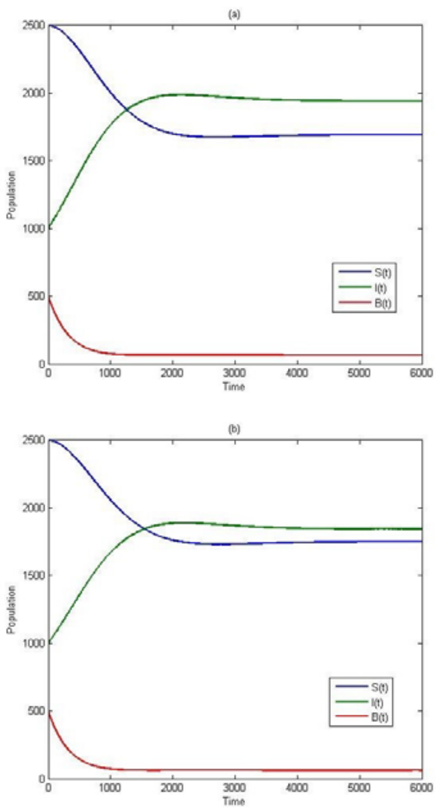


Figure 6: The trajectories of system (1), we use, *A=500*, $\beta_\circ$ =0.001, β_1 =0.0001, β_2 =0.0001, η =0.01, μ_1 =0.1, μ_2 =0.3, m=1, *v*=2, *w*=0.1, *K*=5, δ =0.01, with (a) *r*=0.1, (b) *r*=0.3, (c) *r*=0.5.

Similar results are obtained, as those shown in case of increasing *r*, in case of increasing the recovery rate, that means increasing *m* as shown in the following figures:



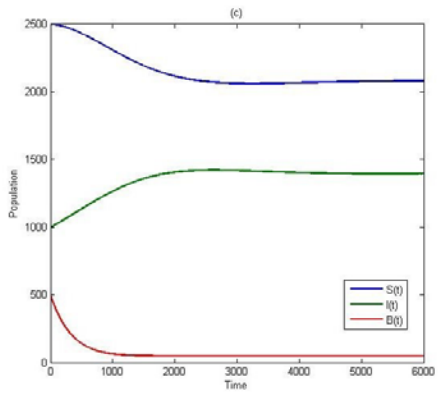


Figure 7: The trajectories of system (1), we use, *A=500*, $\beta_\circ$ =0.001, β_1 =0.0001, β_2 =0.0001, η =0.01, μ_1 =0.1, μ_2 =0.3, *r*=0.05, *v*=2, *w*=0.1, *K*=5, δ =0.01, with (a) *m*=2, (b) *m*=3, (c) *m*=7.

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On T-extending modules

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Abstract

In this paper we introduce the concepts of the T-direct sum and T-extending modules and we give some basic properties of these types of modules. Also we define the relations $\boldsymbol{\alpha}_{\mathbf{T}}$ and β_{T} on the set of submodules containing T of a module M and we give some basic properties.

Keywords: extending modules, T-essential module**,** T-closed modules

1- Introduction

In this paper, all rings are associative with identity and all modules are unitary left R-modules. Recall that a submodule A of an R-module M is essential submodule of M{denoted by $A \leq_e M$}, if for every $B \leq M$, $A \cap B = 0$ implies that $B = 0$.

A submodule B of a module M is called complement for a submodule A of M if it is maximal with respect to the property that $A \cap B = 0$. More details about essential submodules and complement can be found in [1] .

A module M is an **extending module** (denoted by CS- module), if every submodule of M is essential in a direct summand of M , see [2, 3].

Let M be a module. Recall the following relation on the set of submodules of M : **A** $\boldsymbol{\alpha}$ **B** if there exists a submodule C of M such that $A \leq_e C$ and $B \leq_e C$, see [4]. Let M be a module. Recall the following relation on the set of submodules of M: **A** $\boldsymbol{\beta}$ **B** if $A \cap B \leq_e A$ and $A \cap B \leq_e B$, see [4]. In [5], the authors introduced the definition of T–essential (complement) submodules as follows: Let $T \lneqq M$, a submodule A of M is called T–essential submodule of M {denoted by $A \leq_{Tes} M$}, provided that $A \nleq T$ and for each submodule B of M , $A \cap B \leq T$ implies that $B \leq T$. A submodule B of M is called a T –complement for a submodule A in M if B is maximal with respect to the property that $A \cap B \leq T$. In [6], we introduce the definition of T-closed submodules as follows: Let T , A and B be submodules of a module M . A is called a T–closed submodule of M (denoted by $A \leq_{Tc} M$), if $A \leq_{Tes} B$ implies that $A + T = B$, for every submodule B of M.

In section 2 , we will introduce the definition of T-direct sum modules as follows : Let T, A and B be submodules of a module M. M is called T-direct sum of A and B (denoted by $M = A \oplus_T B$). If $M = A + B$ and $A \cap B \leq T$.

In this case, each of A and B is called a T-direct summand of M .We prove that Let T, A and B be submodules of a distributive module M. If B is a T-complement for A in M , then $A \oplus_T B \leq_{Tes} M$, see proposition (2.11). Also we introduce the definition of T-extending modules as follows:

Let T be a submodule of a module M. We say that M is **T-extending module** (denoted by T-CS modules) if every submodule of M which contains T is T-essential in every T-closed submodule of M which contains T is a T-direct summand of M, see proposition (2.15).

In section three , we will define the following relation : Let A and B be submodules of a module M with $T \leq A$ and $T \leq B$. We say that **A** $\boldsymbol{\alpha}_{\mathbf{T}}$ **B** if there exists a submodule C such that $A \leq_{Tes} C$ and $B \leq_{Tes} C$.

Also we define the following relation : Let A and B be submodules of a module M with $T \leq A$ and $T \leq B$. We say that **A** $\boldsymbol{\beta}_{\mathbf{T}}$ **B** if $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. We prove that : The β_T is an equivalence relation , see proposition (3.10).

2. The T-extending modules

In this section , we will introduce the concepts of the **T-direct sum** and **T-extending modules** and we illustrate it by some examples. We also give some basic properties of these type of modules.

Definition (2.1): Let T, A and B be submodules of a module M. M is called **T-direct sum** of A and B (denoted by $M = A \oplus_T B$). If $M = A + B$ and $A \cap B \leq T$. In this case , each of A and B is called a T-direct summand of M .

Let M be a module . Clearly that every direct summand of M is a T-direct summand. And when T = 0, a submodule A of M is a T-direct summand of M if and only if A is a direct summand of M.

Examples (2.2):

(1) Consider the module Z as Z-module and let T = 6Z . Clearly that $Z = 2Z \oplus_T 3Z$. But 2Z is not a direct summand of Z . Now let T = 4Z . $2Z \cap 3Z = 6Z \nleq 4Z$, then Z is not 4Z-direct sum of 2Z and 3Z.

(2) The Z_{12} as Z-module. Let $T = \{\bar{0},\bar{6}\}$, $A = \{\bar{0},\bar{2},\bar{4}, \bar{6},\bar{8},\overline{10}\}$ and $B = \{\bar{0},\bar{3},\bar{6},\bar{9}\}$. One can easily show that A is $\{\bar{0},\bar{6}\}$-direct summand of Z_{12} , and A is not direct summand of Z_{12} .

Proposition (2.3): Let T, A and B be submodules of a module M such that $\frac{M}{T} = \frac{A}{T} \oplus \frac{B}{T}$. Then $M = A \oplus_T B$.

Proof: suppose that $\frac{M}{T} = \frac{A}{T} \oplus \frac{B}{T}$. Then $M = A + B$ and $\frac{A}{T} \cap \frac{B}{T} = \frac{A \cap B}{T} = 0$ and hence $A \cap B = T$. Thus $M = A \oplus_T B$

Note: The converse of proposition is not true in general , for example . Consider the module Z as Z-module and

D is a T-direct summand of M. Thus M is a T-extendin .

Remark (2.16): Let T be a submodule of M. If T = 0 then M is T-extending if and only if M is extending.

Proof: Clear.

let T = A = 4Z , B = 3Z. Cleary that M =A $\oplus_T$ B . But A ∩ B = 12Z ≠ T. Thus $\frac{M}{T}$ is not the direct sum of $\frac{A}{T}$ and $\frac{B}{T}$.

Remark (2.4): Let T , A and B be submodules of a module M such that A ≤ B ≤ M and T ≤ B . If A is a T-direct summand of M , then A is a T-direct summand of B.

Proof: Let A be a T-direct summand of M, then M = A $\oplus_T$ C, for some submodule C of M. Since A ≤ B , then by modular law, B = M ∩ B = (A $\oplus_T$ C) ∩ B = A $\oplus_T$ (C ∩ B). Thus A is a T-direct summand of B.

A module M is called a **distributive module** if A ∩ (B + C) = (A ∩ B) + (A ∩ C) , for all submodules A , B and C of M. See [7].

Lemma (2.5): [8] Let A , B and C be are submodules of a module M . Then the following statement are equivalent :

(1) A ∩ (B + C) = (A ∩ B) + (A ∩ C).

(2) A + (B ∩ C) = (A + B) ∩ (A + C).

Proposition (2.6): Let T, A and B be submodules of a distributive module M such that M = A $\oplus_T$ B , then $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$.

Proof: Assume that M = A $\oplus_T$ B .Then $\frac{M}{T} = \frac{A+B+T}{T} = \frac{A+T}{T} + \frac{B+T}{T}$. Since A ∩ B ≤ T , then (A ∩ B) + T ≤ T . Since M is a distributive module, the (A + T) ∩ (B + T) = (A ∩ B) + T ≤ T, by lemma (2.5). But T ≤ (A + T) ∩ (B + T) , therefore (A + T) ∩ (B + T) = T . Hence $\frac{A+T}{T} \cap \frac{B+T}{T} = 0$. Thus $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$.

Proposition (2.7): Let T , A and B be submodules of a module M such that A ≤ B . If A is T-direct summand of B and B is T-direct summand of M , then A is T-direct summand of M.

Proof: Suppose that A is T-direct summand of B , then B = A $\oplus_T$ C , where C be a submodule of B . Since B is T-direct summand of M, then M = B $\oplus_T$ D , where D be a submodule of M. Implies that M = (A $\oplus_T$ C) $\oplus_T$ D . Hence M = (A + C) + D = A + (C + D) and A ∩ (C ∩ D) = (A ∩ C) ∩ D ≤ T. Then M =A $\oplus_T$ (C $\oplus_T$ D). Thus A is T-direct summand of M.

Proposition (2.8): Let T, A and B be submodules of a distributive module M such that M =A $\oplus_T$ B .Then B + T is a T-complement for A + T in M.

Proof: Suppose that M is a distributive module and M = A $\oplus_T$ B . Then by (2.6) , $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$. Thus B + T is a T-complement for A+T in M, by [9,Coro.3.5,p. 907]

Corollary (2.9): Let T, A and B be submodules of a distributive module M such that M = A $\oplus_T$ B . Then A + T is T-closed submodule of M.

Proof: Assume that M = A$\oplus_T$ B, then $\frac{M}{T} = \frac{A+T}{T} \oplus \frac{B+T}{T}$, by (2.6). Then $\frac{A+T}{T}$ is closed in $\frac{M}{T}$ by [1] . Thus A + T is T-closed submodule of M, by [6 , Prop. 2.9 , p. 1684].

a T-direct summand of M. we prove that : Let M be a module. Then M is T-extending module if and only if.

Proof: Let M = A $\oplus_T$ B, then A is a T-closed in M , by (2.9). Since T ≤ A , then $\frac{A}{T}$ is closed submodule of $\frac{M}{T}$, by [6, Coro. 2.10 , p. 1684].

Proposition (2.11): Let T , A and B be submodules of a distributive module M . If B is a T-complement for A in M , then A $\oplus_T$ B $\leq_{Tes}$ M.

Proof: Let B be a T-complement for A in M, then A ∩ B ≤ T. Let C be a submodule of M such that (A $\oplus_T$ B) ∩ C ≤ T. Since M is a distributive module, then (A∩C) $\oplus_T$ (B∩ C) ≤ T and A ∩ (B $\oplus_T$ C) = (A ∩ B) $\oplus_T$ (A ∩ C) ≤ T. But B is maximal with respect to property that A ∩ B ≤ T, therefore B + C = B. Implies that C ≤ B. Hence C = C ∩ B ≤ T .Thus A $\oplus_T$ B $\leq_{Tes}$ M.

Corollary (2.12): Let T, A and B be submodules of a distributive module M. If $\frac{B}{T}$ is a relative complement for $\frac{A}{T}$ in $\frac{M}{T}$ then A $\oplus_T$ B $\leq_{Tes}$ M.

Proof: Suppose that $\frac{B}{T}$ is a relative complement for $\frac{A}{T}$ in $\frac{M}{T}$, then B is a T-complement for A in M, by [9, Prop. 3.4, p. 907]. Hence A $\oplus_T$ B $\leq_{Tes}$ M, by (2.11).

Proposition (2.13): Let A, B, C and D be submodules of a distributive module M such that T, A, C ≤ B . If M = B $\oplus_T$ D and C is a T-complement of A in B , then C $\oplus_T$ D is a T-complement for A in M.

Proof: Let M = B $\oplus_T$ D and C be a T-complement for A in B. Then M = B + D, B ∩ D ≤ T and A ∩ C ≤ T. Since C ≤ B, then C ∩ D ≤ B ∩ D ≤ T. As A ≤ B, then A ∩ D ≤ B ∩ D ≤ T. But M is a distributive module, hence we obtain A∩ (C $\oplus_T$ D) = (A ∩ C) $\oplus_T$ (A ∩ D) ≤ T .Now let L be a submodule of M such that C$\oplus_T$ D ≤ L and A ∩ L≤ T. Then (L ∩ A) ∩ B = (A ∩ L) ∩ B ≤ T. But C is maximal with respect to the property that A ∩ C ≤ T, therefore, C = L ∩ B. Thus L = M ∩ L = (B $\oplus_T$ D) ∩ L = (B ∩ L) $\oplus_T$ (D ∩ L) = C $\oplus_T$ D. Which means C $\oplus_T$ D is a T-complement for A in M.

We introduce the following definition

Definition (2.14): Let T be a submodule of a module M. We say that M is **T-extending module** (denoted by T-CS modules) if every submodule of M which contains T is T-essential in a T-direct summand of M.

Proposition (2.15): Let M be a module. Then M is T-extending module if and only if every T-closed submodule of M which contains T is a T-direct summand of M.

Proof: Suppose that M is a T-extending module and let A be a T-closed submodule of M such that T ≤ A . Since M is a T-extending module , then there exist a T-direct summand D of M such that A $\leq_{Tes}$ D . But A is a T-closed submodule of M , therefore A + T = D .Thus A = D.

Conversely, let A be a submodule of M such that T ≤ A . So there exist a T-closed submodule D in M such that A $\leq_{Tes}$ D, by [6, Prop. 2.12, P.1684]. By our assumption

Since Z_{P^∞} is a uniserial module , then either $A \leq B$ or $B \leq A$. If $A \leq B$, we get $A \cap B = A \leq T$ which is a contradiction . Thus $B \leq A$ and hence $A \cap B = B \leq T$. Thus A is T-essential submodule of Z_{P^∞}. By the same way D is T-essential in Z_{P^∞} then A and D are T-essential in Z_{P^∞}. Thus $Z_{p^n}\ \alpha_T\ Z_{p^m}$.

Proposition (2.17): Every T-direct summand contain T of a distributive and T-extending module is T-extending module.

Proof: Let M be a distributive and T-extending module such that $M = A \oplus_T B$ and $T \leq A$, where T , A and B are submodules of M. Let C be a T-closed submodule in A such that $T \leq C$. Since A is a T-direct summand of M, then A is a T-closed submodule of M, by (2.9). Thus C is a T-closed in M , by [6, Th. 2.14 , p. 1684]. But M is a T-extending , therefore C is a T-direct summand of M by (2.15). Since $C \leq A$, then C is a T-direct summand of A , by (2.4).

Examples (2.18):

(1) Consider the module Z_6 as Z-module and let $T =\{\bar{0},\bar{2},\bar{4}\}$. Then Z_6 and $\{\bar{0},\bar{2},\bar{4}\}$ are they only submodules of Z_6 that containing T. Since $\{\bar{0},\bar{2},\bar{4}\}$ is a T-essential and a T-direct summand of Z_6 and Z_6 is a T-essential of Z_6 .Then Z_6 is T-extending module.

(2) Consider the module Z as Z-module . Let T = 2Z , then Z and 2Z are they only submodules of Z that containing T . Since 2Z is T-essential in 2Z and 2Z is T-direct summand of Z .Then Z is 2Z-extending module.

(3) The module $M = Z_8 \oplus Z_2$ as a Z-module . It's known that M is not extending module , by [10, ex. (2.4.18). Ch.2] . Hence M is not {0}-extending module. Now let $T = \{\bar{0},\bar{2},\bar{4},\bar{6}\} \oplus Z_2$. Since M and T are the only submodules that containing T , then one can easily check that M is a T-extending module.

Proposition (2.19): Let T be a submodule of a module M. If $\frac{M}{T}$ is extending module, then M is a T-extending module.The converse is true if M is a distributive module.

Proof: Let A is a submodule of M such that $T \leq A$. Since $\frac{M}{T}$ is an extending module , then there exist a direct summand $\frac{B}{T}$ of $\frac{M}{T}$ such that $\frac{A}{T} \leq_e \frac{B}{T}$ .Therefore $A \leq_{Tes} B$ by [5,Lem. 2.3, P. 17] and B is a T-direct summand of M, by (2.3). Thus M is a T-extending .

For the converse , Let M be a distributive module $\frac{A}{T}$ be a submodule of $\frac{M}{T}$. Since M is T-extending and A is a submodule of M , then there exist a T-direct summand B of M such that $A \leq_{Tes} B$. Thus $\frac{A}{T} \leq_e \frac{B}{T}$, by [5, Lem. 2.3, p. 17]. Hence $M = B \oplus_T B_1$, for some submodule B_1 of M . But M is a distributive module , therefore $\frac{M}{T} = \frac{B}{T} \oplus \frac{B1+T}{T}$, by proposition (2.6). So $\frac{B}{T}$ is a direct summand of $\frac{M}{T}$. Thus $\frac{M}{T}$ is extending.

Theorem (2.20): Let T and A be submodules of a T-extending module M such that $T \leq A$. If the intersection of A with any T-direct summand of M containing T is a T-direct summand of A then A is T-extending module.

Proof: Let M be a T-extending module and N be a submodule of A such that $T \leq N$, then there exist T-direct

Corollary (2.10): Let T , A and B be submodules of adistributive module M such that $T \leq A$ and $M = A \oplus_T B$. Then $\frac{A}{T}$ is a closed submodule of $\frac{M}{T}$. K of M which containing T and either $K \cap A \leq T$ or $K \cap B \leq T$ is a T-direct summand of M.

Proof: Assume K is T-closed of M such that $T \leq K$ and $T = K \cap A$. Since M is T-extending module , then K is a T-direct summand of M by , (2.15) .

Theorem (2.22): Let T , A and B be submodules of a module M such that $M = A \oplus_T B$ and $T \leq A \cap B$. If every T-closed submodule K of M which containing T and either $K \cap A \leq T$ or $K \cap B \leq T$ is a T-direct summand of M , then every T-complement containing T for A or B in M is T-direct summand of M and T-extending module .

Proof: Let K be a T-complement for A in M such that $T \leq K$. Then K is a T-closed submodule in M . by [6. Th. 2.18 , P. 1684]. But $K \cap A \leq T$, therefore by our a assumption K is a T-direct summand of M.

Let L be a T-closed submodule of K such that $T \leq L$. Then L is a T-closed in M , by [6. Th.2.14 , p. 1684]. Since $L \cap A \leq K \cap A \leq T$. Then by our assumption L is a T-direct summand of M and hence L is a T-direct summand of K , by (2.4) . Thus K is a T-extending of M.

Theorem (2.23): Let T , A and B be submodules of a module M such that $M = A \oplus_T B$ and $T \leq A \cap B$. If M is T-extending module, then every T-complement containing T for A or B in M is T-direct summand of M and T-extending module .

Proof: Suppose that M is T-extending module and let K is a T-complement for A in M contain T , then K is T-closed in M, by [6. Th. 2.18 , P. 1684]. Since $K \cap A \leq T$, then K is a T-direct summand of M , by (2.21). Thus K is T-extending module , by (2.22).

3- The relations $\boldsymbol{\alpha_T}$ and $\boldsymbol{\beta_T}$:

In this section we define the relations $\boldsymbol{\alpha_T}$ and β_T . Also we give some basic properties of these relations.

Definition (3.1): Let T be a submodule of a module M and let S_T be the set of submodules of M that containing T. Let A and $B \in S_T$. We say A $\boldsymbol{\alpha_T}$ B if there exists a submodule C such that $A \leq_{Tes} C$ and $B \leq_{Tes} C$.

Let M be a module and T = 0 . Then one can easily show that A α B if and only if A α_T B , for each submodules A and B of M.

Examples (3.2):

(1) The module Z_4 as Z-module. Let $T =\{\bar{0}, \bar{2}\}$, $A =\{\bar{0}, \bar{2}\}$ and $B = Z_4$. Since A and B are T-essential in Z_4, then A α_T B.

(2) The module Z_{12} as Z-module . Let $T = \{\bar{0},\bar{6}\}$, $A = \{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ and $B = Z_{12}$. Since B is T-essential in Z_{12} and clearly that $A \cap \{\bar{0},\bar{3},\bar{6},9\} = T$. But $\{\bar{0},\bar{3},\bar{6},9\} \nleq T$, therefore A is not T-essential in Z_{12} . Thus A is not relate to B by α_T

(3) The module Z_{P^∞} as Z-module. Let $T = (\frac{1}{pn} + Z)$, $A = (\frac{1}{pm} + Z)$ and $D = (\frac{1}{pr}+ Z)$, where n, m, r ∈ Z and m, r > n .Let B be a submodule of Z_{P^∞} such that $A \cap B \leq T$.

$A \cap C \le T$ implies $B \cap C \le T$ and $B \cap D \le T$ implies $A \cap D \le T$, for each submodules C and D of M.

Proposition (3.3): Let T be a submodule of a module M. Then $A\ \alpha_T\ B$ if and only if $\frac{A}{T}\ \alpha\ \frac{B}{T}$, for each A and B $\in S_T$.

Proof: Let $A\ \alpha_T\ B$. Then there exists a submodule C of M such that $A \le_{Tes} C$ and $B \le_{Tes} C$. Then $\frac{A}{T} \le_e \frac{C}{T}$ and $\frac{B}{T} \le_e \frac{C}{T}$, by [5,Lem. 2.3, P. 17].. Thus $\frac{A}{T}\ \alpha\ \frac{B}{T}$.

Conversely , let $\frac{A}{T}\ \alpha\ \frac{B}{T}$, then there exists a submodule $\frac{C}{T}$ of $\frac{M}{T}$ such that $\frac{A}{T} \le_e \frac{C}{T}$ and $\frac{B}{T} \le_e \frac{C}{T}$. Then $A \le_{Tes} C$ and $B \le_{Tes} C$, by [5, Lem. 2.3, P. 17]. Thus $A\ \alpha_T\ B$.

Remark(3.4): The α_T is a reflexive and symmetric relation.

Proof: Clear.

Proposition (3.5): Let T be a submodule of a module M. Then M is T-extending if and only if for each submodule $A \in S_T$, there exists a T-direct summand $D \in S_T$ such that $A\ \alpha_T\ D$

Proof: ⇒) Suppose that M is T-extending , and let $A \in S_T$. Since M is T-extending , then there exists a T-direct summand $D \in S_T$ such that $A \le_{Tes} D$, we want to show that there exists a submodule B of M such that $A \le_{Tes} B$ and $D \le_{Tes} B$. Let B = D , then $A \le_{Tes} D$ and $D \le_{Tes} D$. Thus $A\ \alpha_T\ D$.

⇐) Let $A \in S_T$, by our assumption , there exists a T-direct summand $D \in S_T$ such that $A\ \alpha_T\ D$. Thus there exists a submodule $B \in S_T$ such that $A \le_{Tes} B$ and $D \le_{Tes} B$. It is enough to show that B is a T-direct summand of M . Let $M = D \oplus_T D_1$, where D_1 is a submodule of M . Since $D \le B$ then $M = B + D_1$. Since $D \cap D_1 \le T$, then $(B \cap D) \cap D_1 \le T$. But $D \le_{Tes} B$, therefore $D_1 \cap B \le T$. Hence $M = B \oplus_T D_1$. Claim that B = D . To show that , Let $b \in B$, then $b = d + d_1$, where $d \in D$ and $d_1 \in D_1$. So $b - d = d_1 \in (B \cap D_1) \le T \le D$ .Hence $b = d + d_1 \in D$. This implies that B = D . Thus M is a T-extending module .

Proposition (3.6): Let T, A and B be submodules of a module M such that A and $B \in S_T$ .If $A\ \alpha_T\ B$, then there exists a submodule C of M such that $\frac{C}{A}$ and $\frac{C}{B}$ are singular .

Proof: Assume that $A\ \alpha_T\ B$, then there exists a submodule $C \in S_T$ such that $A \le_{Tes} C$ and $B \le_{Tes} C$. Hence $\frac{A}{T} \le_e \frac{C}{T}$ and $\frac{B}{T} \le_e \frac{C}{T}$, by [5, Lem. 2.3, P. 17]. Now consider the following two short exact sequences:

$$0 \rightarrow \frac{A}{T} \xrightarrow{i} \frac{C}{T} \xrightarrow{\pi 1} \frac{C/T}{A/T} \rightarrow 0$$

$$0 \rightarrow \frac{B}{T} \xrightarrow{j} \frac{C}{T} \xrightarrow{\pi 2} \frac{C/T}{B/T} \rightarrow 0$$

Where i , j are inclusion map and π_1 , π_2 are the natural epimorphisms. Since $\frac{A}{T} \le_e \frac{C}{T}$ and $\frac{B}{T} \le_e \frac{C}{T}$, then $\frac{C/T}{A/T}$ and $\frac{C/T}{B/T}$ are singular , by [2, Prop.1.20, P.31] . By the third isomorphis theorem , $\frac{C/T}{A/T} \cong \frac{C}{A}$ and $\frac{C/T}{B/T} \cong \frac{C}{B}$. Thus $\frac{C}{A}$ and $\frac{C}{B}$ are singular.

Definition (3.7): Let T be a submodule of a module M and let A and $B \in S_T$, then we say that $A\ \boldsymbol{\beta_T}\ B$ if $A \cap B \le_{Tes} A$ and $A \cap B \le_{Tes} B$.

Proof:⇒) Suppose that $A\ \beta_T\ B$ and let C be a submodule of M such that $A \cap C \le T$ .Then $A \cap B \cap C \le T$, hence $(A \cap B) \cap (B \cap C) \le T$. But $A \cap B \le_{Tes} B$,therefore

summand D of M such that $T \le D$ and $N \le_{Tes} D$. Since $N \le A \cap D$, then $N \le_{Tes} A \cap D$, by [5, Prop. 2.12 , P. 19]. By our $A \cap D$ is a T-direct summand of A . Thus A is T-extending module .

Theorem (2.21): Let T , A and B be submodules of a module M such that $M = A \oplus_T B$ and $T \le A \cap B$. If M is T-extending module , if every T-closed submodule

Examples (3.8):

(1) The module Z_4 as Z-module . Let $T = \{\bar{0}\}$,$A = \{\bar{0},\bar{2}\}$ and $B = Z_4$, then $A \cap B = A \le_{Tes} A$ and $A \cap B = A \le_{Tes} B$. Thus $\{\bar{0},\bar{2}\}\ \beta_T\ Z_4$.

(2) Consider the module $Z_{P\infty}$ as Z-module . Let $T = (\frac{1}{pn} + Z)$, $A = (\frac{1}{pm} + Z)$ and where n , m ∈ Z and m > n and let $B = Z_P{}^{\infty}$. Since by (3.2-3), $A \le_{Tes} Z_P{}^{\infty}$. and $A \le_{Tes} A$ .Then $Z_p{}^{m}\ \beta_T\ Z_P{}^{\infty}$.

(3) The module Z_{12} as Z-module. Let $T = \{\bar{0},\bar{6}\}$, $A = \{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ and $B = Z_{12}$,then $A \cap B = A \le_{Tes} A$. But A is not T-essential in Z_{12} , by (3.2-2). Therefore $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ is not related to Z_{12} by β_T .

Properties (3.9): Let T , A and B be a submodules of a module M such that A , $B \in S_T$. Then $A\ \beta_T\ B$ if and only if $\frac{A}{T}\ \beta\ \frac{B}{T}$

Proof: ⇒) Suppose that $A\ \beta_T\ B$, then $A \cap B \le_{Tes} A$ and $A \cap B \le_{Tes} B$. Then $\frac{A \cap B}{T} \le_e \frac{A}{T}$ and $\frac{A \cap B}{T} \le_e \frac{B}{T}$, by [5, Lem. 2.3, p. 17]. Thus $\frac{A}{T}\ \beta\ \frac{B}{T}$.

⇐) Let $\frac{A}{T}\ \beta\ \frac{B}{T}$, then $\frac{A \cap B}{T} \le_e \frac{A}{T}$ and $\frac{A \cap B}{T} \le_e \frac{B}{T}$. Then $A \cap B \le_{Tes} A$ and $A \cap B \le_{Tes} B$, by [5, Lem. 2.3, P. 17]. Thus $A\ \beta_T\ B$.

Proposition (3.10): The β_T is an equivalence relation.

Proof: Clearly that β_T is reflexive and symmetric . We want to show β_T is transitive , let A , B and $C \in S_T$ such that $A\ \beta_T\ B$ and $B\ \beta_T\ C$. Since $A\ \beta_T\ B$ and $B\ \beta_T\ C$, then $A \cap B \le_{Tes} A$, $A \cap B \le_{Tes} B$, $B \cap C \le_{Tes} B$ and $B \cap C \le_{Tes} C$. Let L be a submodule of A such that $(A \cap C) \cap L \le T$, then $(B \cap C) \cap (A \cap B \cap L) \le T$. Since $B \cap C \le_{Tes} B$, then $A \cap B \cap L \le T$. Hence $(A \cap B) \cap (A \cap L) \le T$. But $A \cap B \le_{Tes} A$, therefore $A \cap L \le T$. Since $L \le A$, then $L \le T$. So$A \cap C \le_{Tes} A$. Similarly $A \cap C \le_{Tes} C$. Thus β_T is an equivalence relation.

Proposition (3.11): Let T , A and B be submodules of a module M such that A and $B \in S_T$ .Then $A\ \beta_T\ B$ if and only if

$B \cap C \leq T$. Now let $B \cap D \leq T$,where D is a submodule of M . Then $A \cap B \cap D \leq T$ and hence $(A \cap B) \cap (A \cap D) \leq T$. But$A \cap B \leq_{Tes} A$, therefore $A \cap D \leq T$.

$\Leftarrow$) To show $A\ \beta_T\ B$. Let L be a submodule of A such that $A \cap B \cap L \leq T$ .Since $A \cap (B \cap L) \leq T$, then by our assumption $B \cap L = B \cap (B \cap L) \leq T$. Hence $A \cap L \leq T$. But $L \leq A$, therefore $L \leq T$. Similarly , let K be a Submodule of B such that $(A \cap B) \cap K \leq T$. Then by our assumption $A \cap K = A \cap (A \cap K) \leq T$. Hence $B \cap K \leq T$. But $K \leq B$, therefore $K \leq T$. Thus $A\ \beta_T\ B$.

Proposition (3.12): Let T, A and B be submodules of a module M such that A and $B \in S_T$. Then $A\ \beta_T\ B$ if and only if for each $x \in A - T$, $y \in B - T$ there exists $r , r_1 \in R$ such that $rx \in B - T$ and $r_1 y \in A - T$.

Proof: Assume that $A\ \beta_T\ B$, then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Hence for each $x \in A - T$ there exists $r \in R$ such that $rx \in (A \cap B) - T$. Thus $rx \in B - T$. Similarly , for each $y \in B - T$ there exists $r_1 \in R$ such that $r_1 y \in (A \cap B) - T$ and hence $r_1 y \in A - T$.

For the converse, assume that $x \in A - T$. Then there exists $r \in R$ such that $rx \in B - T$. So $rx \in (A \cap B) - T$. Thus $A \cap B \leq_{Tes} A$. Now let $y \in B - T$, then there exists $r_1 \in R$ such that $r_1 y \in A - T$. Hence $r_1 y \in (A \cap B) - T$. So $A \cap B \leq_{Tes} B$. Thus $A\ \beta_T\ B$.

Proposition (3.13): Let T , A_1 , A_2 , B_1 and B_2 be submodules of a module M such that A_1 ,A_2 , B_1 and $B_2 \in S_T$. If $A_1\ \beta_T\ B_1$ and $A_2\ \beta_T\ B_2$, then $(A_1 \cap A_2)\ \beta_T\ (B_1 \cap B_2)$.

Proof: Suppose that $A_1\ \beta_T\ B_1$ and $A_2\ \beta_T\ B_2$. Then $A_1 \cap B_1 \leq_{Tes} A_1$, $A_1 \cap B_1 \leq_{Tes} B_1$, $A_2 \cap B_2 \leq_{Tes} A_2$ and $A_2 \cap B_2 \leq_{Tes} B_2$. Hence $(A_1 \cap A_2) \cap (B_1 \cap B_2) \leq_{Tes} A_1 \cap A_2$ and $(A_1 \cap A_2) \cap (B_1 \cap B_2) \leq_{Tes} B_1 \cap B_2$, by [9, Prop.2.6 . P. 903]. Hence $(A_1 \cap A_2)\ \beta_T\ (B_1 \cap B_2)$.

Proposition (3.14): Let $f : M \to N$ be an epimorphism and T , A , B be submodules of N such that A and $B \in S_T$. If $A\ \beta_T\ B$, then $f^{-1}(A)\ \beta_{f^{-1}(T)}\ f^{-1}(B)$.

Proof: Let A and B be submodules of N such that $A\ \beta_T\ B$, then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Hence by [5, Lem. 2.15, P. 20] , $f^{-1}(A \cap B) \leq_{f^{-1}(T)es} f^{-1}(A)$, implies that $f^{-1}(A) \cap f^{-1}(B) \leq_{f^{-1}(T)es} f^{-1}(A)$. Since $A \cap B \leq_{Tes} B$, then by [5, Lem. 2.15, P. 20], $f^{-1}(A \cap B) \leq_{f^{-1}(T)es} f^{-1}(B)$, implies that $f^{-1}(A) \cap f^{-1}(B) \leq_{f^{-1}(T)es} f^{-1}(B)$. Thus $f^{-1}(A)\ \beta_T\ f^{-1}(B)$.

Proposition (3.15): Let T , A and B be submodules of a module M such that A and $B \in S_T$. If $A\ \beta_T\ B$, then $\frac{A}{A \cap B}$ and $\frac{B}{A \cap B}$ are singular.

Proof: Assume that $A\ \beta_T\ B$. Then $A \cap B \leq_{Tes} A$ and $A \cap B \leq_{Tes} B$. Then $\frac{A \cap B}{T} \leq_e \frac{A}{T}$ and $\frac{A \cap B}{T} \leq_e \frac{B}{T}$, by [5, Lem. 2.3, P. 17].

Now consider the following two short exact sequences:

$$0 \to \frac{A \cap B}{T} \xrightarrow{i} \frac{A}{T} \xrightarrow{\pi_1} \frac{A/T}{(A \cap B)/T} \to 0$$

$$0 \to \frac{A \cap B}{T} \xrightarrow{j} \frac{B}{T} \xrightarrow{\pi_2} \frac{B/T}{(A \cap B)/T} \to 0$$

where i , j are inclusion map and π_1 , π_2 are the natural epimorphisms . Since $\frac{A \cap B}{T} \leq_e \frac{A}{T}$ and $\frac{A \cap B}{T} \leq_e \frac{B}{T}$, then $\frac{A/T}{(A \cap B)/T}$ and $\frac{B/T}{(A \cap B)/T}$ are singular , by [2, Prop.1.20, P.31] . Hence by the third isomorphism theorem, $\frac{A/T}{(A \cap B)/T} \cong \frac{A}{A \cap B}$ and $\frac{B/T}{(A \cap B)/T} \cong \frac{A}{A \cap B}$. Thus $\frac{A}{A \cap B}$ and $\frac{B}{A \cap B}$ are singular.

Corollary (3.16): Let T , A and B be submodules of a module M such that A and $B \in S_T$. If $A\ \beta_T\ B$, then $\frac{A+B}{A}$ and $\frac{A+B}{B}$ are singular.

Proof: Clear by the second isomorphism theorem.

Proposition (3.17): Let $\{M_\alpha\}_{\alpha \in \Lambda}$ be a family of modules and T_α , A_α and B_α be submodules of M_α , for each $\alpha \in \Lambda$ such that $T_\alpha \leq A_\alpha \cap B_\alpha$. If $A_\alpha\ \beta_{T\alpha}\ B_\alpha$, for each $\alpha \in \Lambda$, then $(\oplus_{\alpha \in \Lambda} A_\alpha)\ \beta_{\oplus_{\alpha \in \Lambda} T\alpha}\ (\oplus_{\alpha \in \Lambda} B_\alpha)$.

Proof : Let $A_\alpha\ \beta_{T\alpha}\ B_\alpha$ for each $\alpha \in \Lambda$,then $A_\alpha \cap B_\alpha \leq_{(T\alpha)es} A_\alpha$ and $A_\alpha \cap B_\alpha \leq_{(T\alpha)es} B_\alpha$. Hence by [5, Lem. 2.3, P. 17] . $\frac{A\alpha \cap B\alpha}{T\alpha} \leq_e \frac{A\alpha}{T\alpha}$ and $\frac{A\alpha \cap B\alpha}{T\alpha} \leq_e \frac{B\alpha}{T\alpha}$. Then by [1], $\oplus_{\alpha \in \Lambda} \frac{A\alpha \cap B\alpha}{T\alpha} \leq_e \oplus_{\alpha \in \Lambda} \frac{A\alpha}{T\alpha}$ and $\oplus_{\alpha \in \Lambda} \frac{A\alpha \cap B\alpha}{T\alpha} \leq_e \oplus_{\alpha \in \Lambda} \frac{B\alpha}{T\alpha}$. But $\oplus_{\alpha \in \Lambda} \frac{A\alpha \cap B\alpha}{T\alpha} \cong \frac{\oplus_{\alpha \in \Lambda}(A\alpha \cap B\alpha)}{\oplus_{\alpha \in \Lambda} T\alpha}$, $\oplus_{\alpha \in \Lambda} \frac{A\alpha}{T\alpha} \cong \frac{\oplus_{\alpha \in \Lambda} A\alpha}{\oplus_{\alpha \in \Lambda} T\alpha}$ and $\oplus_{\alpha \in \Lambda} \frac{B\alpha}{T\alpha} \cong \frac{\oplus_{\alpha \in \Lambda} B\alpha}{\oplus_{\alpha \in \Lambda} T\alpha}$, therefore $\frac{\oplus_{\alpha \in \Lambda}(A\alpha \cap B\alpha)}{\oplus_{\alpha \in \Lambda} T\alpha} \leq_e \frac{\oplus_{\alpha \in \Lambda} A\alpha}{\oplus_{\alpha \in \Lambda} T\alpha}$ and $\frac{\oplus_{\alpha \in \Lambda}(A\alpha \cap B\alpha)}{\oplus_{\alpha \in \Lambda} T\alpha} \leq_e \frac{\oplus_{\alpha \in \Lambda} B\alpha}{\oplus_{\alpha \in \Lambda} T\alpha}$. Then by [5, Lem. 2.3, P. 17], $\oplus_{\alpha \in \Lambda} (A_\alpha \cap B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T\alpha)es} \oplus_{\alpha \in \Lambda} A_\alpha$ and hence $\oplus_{\alpha \in \Lambda} (A_\alpha \cap B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T\alpha)es} \oplus_{\alpha \in \Lambda} B_\alpha$. Hence $(\oplus_{\alpha \in \Lambda} A_\alpha) \cap (\oplus_{\alpha \in \Lambda} B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T\alpha)es} \oplus_{\alpha \in \Lambda} A_\alpha$ and $(\oplus_{\alpha \in \Lambda} A_\alpha) \cap (\oplus_{\alpha \in \Lambda} B_\alpha) \leq_{(\oplus_{\alpha \in \Lambda} T\alpha)es} \oplus_{\alpha \in \Lambda} B_\alpha$. Thus $(\oplus_{\alpha \in \Lambda} A_\alpha)\ \beta_{\oplus_{\alpha \in \Lambda} T\alpha}\ (\oplus_{\alpha \in \Lambda} B\alpha)$.

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Modelling and Simulation of Airflow in an Inclined Bifurcated Trachea

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***Abstract*:** The effect of horizontal sleeping position on the health of some patients with breathing problems still needs to be clarified. A new update mathematical model for simulating the unsteady airflow inside a bifurcated trachea for various Reynolds numbers and inclination angles is determined. The governing unsteady equations of motion, consisting of two-dimensional Navier-Stokes equations, nonlinear and non-homogenous are derived and numerically solved using the finite difference Marker and Cell (MAC) method. A numerical code based on Matlab platform is developed to calculate specifically, in addition to other flow characteristics, the pressure distribution and the streamlines which are missing in most previous works in this area. The results for axial velocities at a horizontal situation show good agreement with both numerical and other experimental findings. New results show that an increase in the inclination angle diminishes the pressure drop inside the main and a bifurcated trachea, Sleeping in a horizontal position leads to a negative effect for many patients. Consequently, the bed should be positioned at the angle between 30^o and 45^o. The excellent features of these results suggest that the proposed model-based procedure may contribute towards the development of more accurate and effective inclined bed therapy (IBT).

***Keywords*:** Numerical simulation; bifurcated trachea; pressure correction; Inclined Bed Therapy (IBT)

1. Introduction

Mathematical models of a bifurcated trachea are essential for the development of biomedical engineering. Currently, the understanding of airflow through human airways is gaining much research attention either from a numerical viewpoint or from the experimental design. Recently, comprehensive understanding and prediction of phenomena in physiology demanded and life sciences accurate mathematical models with numerical simulation methods. This is particularly true for realizing (IBT) inclined bed therapy, which is a perfect normal remedy for many health problems without using harmful chemicals or any substances in the patients' body. IBT is a new therapy presented by Andrew Fletcher [1] seemed promising. IBT therapy expands the capacity of the body to perform without externally infused of synthetic chemicals. It is beneficial for numerous illnesses linked with breathing problems such as snoring, asthma, mild sleep apnea and chronic obstructive pulmonary disease (COPD) [2-4]. The inspiratory flow rates in the human respiratory system depend on the strength of physical activity. The range of Reynolds numbers (*Re*) of airflow in human trachea range from 800-9300 depend on quiet or heavy breathing [5]. Numerically simulated of respiratory flow patterns introduced in [6] to study the inhalation and the exhalation through a single bifurcation for Reynolds numbers 50 - 4500. Calay *et al*. [7] introduced a numerically simulated of respiratory flow patterns through the trachea and main bronchi at resting with *Re* = 1750 and at maximal exercising with *Re* = 4600. Definitely, the understanding of airflow in human trachea is an alternative method to support the treatment of patients suffering from breathing complications. Thus, a mathematical model depicting the dynamics of airflow movement in the trachea in terms of governing equations of motion is valuable. many computational fluid dynamics researchers to investigate the air flow through the trachea have been carried out. Li *et al*. [8, 9] investigated steady laminar and transient air flow field and particle deposition in a trachea with *Re* = 1201 for breathing in resting conditions. The numerical results of velocities are compared with experimental results of Zhao and Lieber [10]. Their mathematical model with steady, laminar, incompressible, and three-dimensional airflow in rigid airway was developed. Commercial software based on finite volume used to simulate their model. Liu *et al*. [11] utilized a child model to investigate the impacts of physiological features on the airflow patterns and nanoparticle deposition in the upper respiratory tracts. Their model in three-dimensions involved the mouth cavity, larynx, pharynx, trachea and bronchial and it considered to be incompressible, laminar and steady with a low Reynolds number. A mathematical model of airflow in the upper respiratory tract described in [12, 13] considering the air as incompressible and Newtonian with constant density and viscosity. The results obtained using a finite element analysis and (COMSOL software) then the simulation results compared with those in [12] from an analytical calculation based on Fourier series. Alnussairy *et al*. [14, 15] investigated the inclination angle dependence on the unsteady airflow in the main trachea by developing a 2D mathematical model (channel and tube) using two methods analytically and numerically. The exact and numerical solutions are achieved using Bromwich integral and MAC method. Their results for axial velocity at horizontal position of the trachea ($\theta = 0^o$) is compared with the observation of Kongnuan and Pholuang [8] and Zhao and Lieber [6] respectively. Chen *et al.* [16] carried out experiments and simulation to investigate fiber deposition in a single horizontal bifurcation under different steady inhalation conditions. The flow was applied incompressible, Newtonian, laminar and fully developed, a parabolic velocity distribution at the inlet. The governing equations for their model solved using Fluent software. Many researches [7, 8, 9, 11, 13, 16] were achieved using commercial simulation software, and using a given inlet velocity and pressure with proper boundary conditions. These software programs are often very expensive and not easily available. Moreover, the pressure distribution in the tracheal segment is unknown parameter needs an appropriate method for calculation. To avoid this limitation, following [17], [18] and [19]. Yet, no systematic mathematical model is developed to simulate precisely the inclination angle dependent unsteady air flow inside the human trachea. No one investigated the effect of the inclination angle on the airflow in the trachea and main bronchi although the inclined angle position is one of the most important parameters that affect such flow.

This paper investigates the effect of inclination angle position on the airflow pattren in trachea and main bronchi under resting and normal breathing conditions using Marker and cell method, which is helpful because the pressure boundary conditions at the inlet and outlet are not needed. The velocity vector is identified and the results are achieved with the desired degree of accuracy.

2. Governing Equations

Consider air flow model in the tracheal lumen is treated as 2D unsteady, nonlinear, incompressible (low Mach number, M = 0.1, [7], laminar, Newtonian fluid with constant a kinematic viscosity $\nu = \mu/\rho$. The governing momentum and continuity, conservation equations in dimensionless of the axisymmetric air flow in the cylindrical polar coordinate system (r, z) are written as:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(w)=0 \tag{1}$$

$$\frac{\partial w}{\partial t}+\frac{\partial (wu)}{\partial r}+\frac{\partial w^2}{\partial z}+\frac{wu}{r}= \frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{1}{r}\frac{\partial w}{\partial r}(r\frac{\partial w}{\partial r})+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\sin(\theta)}{Fr} \tag{2}$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial r}+\frac{\partial (uw)}{\partial z}+\frac{u^2}{r}= -\frac{\partial p}{\partial r}+\frac{1}{\mathrm{Re}}\left(\frac{1}{r}\frac{\partial u}{\partial r}(r\frac{\partial u}{\partial r})-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\cos(\theta)}{Fr} \tag{3}$$

where $Re = \rho U_0 R_0/\mu$, $Fr = U_0/\sqrt{gR_0}$, $t, \rho, \mu, p, U_0, R_0$ is the Reynolds number, the Froud number, the time, density, viscosity of air, pressure, average of the velocity at the inlet and the radius of the trachea respectively. The axial and radial equations of momentum (2), (3) are imposed with a gravitational force parameter g . The angle θ is the slope between the horizontal direction z and the direction of the trachea [15]. The functions $R_1(z)$ and $R_2(z)$ which represent the outer and inner wall of the trachea respectively [cf. Fig.1] are given by:

$$R_1(z)=\begin{cases} 1, & 0 \le z < z_1 \\ 1+r_1-\sqrt{r_1^2-(z-z_1)^2}, & z_1 \le z < z_2 \\ 2r_1^{'}\sec\beta+(z-z_2)\tan\beta, & z_2 \le z < z_{\max} \end{cases} \tag{4}$$

$$R_2(z)=\begin{cases} 0, & 0 \le z < z_3 \\ \sqrt{r_2^2-(z-z_3-r_2)^2}, & z_3 \le z < z_3+r_2(1-\sin\beta) \\ r_2\cos\beta+z_4, & z_3+r_2(1-\sin\beta) \le z < z_{\max} \end{cases} \tag{5}$$

Where

$r_1=\dfrac{(1-2r_1^{'}\sec\beta)}{(\cos\beta-1)}$; $r_2=\dfrac{(z_3-z_2)\sin\beta}{1-\sin\beta}$;

$z_2=z_1+(1-2r_1^{'}\sec\beta)\dfrac{\sin\beta}{\cos\beta-1}$;

$z_3 = z_2 + 0.5$; $z_4=(z-(z_3+r_2(1-\sin\beta)))\tan\beta$

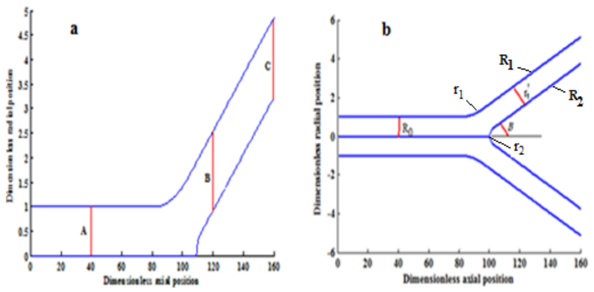

Figure 1. (a) Geometry for a single bifurcated trachea (b) Axisymmetric geometry for a bifurcated trachea

2.1 Initial and Boundary Conditions

The velocity components of airflow stream on the trachea wall should be zero at the rigid wall (no-slip condition). A maximum velocity of airflow is assumed to be fully developed parabolic velocity profile with at the inlet corresponds to the Poiseuille flow [15, 16, 20] of the tracheal lumen yields:

$$w(r,z,t)=U_{\max}\left(1-\left(\frac{r}{R}\right)^2\right), u(r,z,t)=0 \tag{6}$$

at $z=0$, where $U_{\max}=2U_0$

The boundary and initial conditions of the problem is set as:

$$w(r,z,t)=0=u(r,z,t) \text{ on } r=R_1(z) \text{ and } r=R_2(z), z_3 \le z \le z_{\max} \tag{7}$$

$$\frac{\partial w(r,z,t)}{\partial r}=0, \text{ on } r=0,\ 0\le z\le z_3 \tag{8}$$

$$w(r,z,0)=0,\ u(r,z,0)=0,\ p(r,z,0)=0 \text{ for } z>0 \tag{9}$$

2.2 Radial Transformation

The radical transformation is introduced:

$$\xi=\frac{r-R_2(z)}{R(z)}, R(z)\neq 0 , \tag{10}$$

where $R(z)=R_1(z)-R_2(z)$, which has the influence of immobilizing the tracheal wall in the transformed coordinate ξ. Using the radial transformation in equation (10). Therefore, the Equations (1)-(3) takes the form:

$$(\xi R+R_2)\frac{\partial w}{\partial z}-\frac{\xi R+R_2}{R}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)\frac{\partial w}{\partial \xi}+\frac{\partial(u(\xi R+R_2)/R)}{\partial \xi}=0 \tag{11}$$

$$\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\frac{1}{R}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)\frac{\partial p}{\partial \xi}+\text{Conw}+\frac{1}{\text{Re}}\text{Diffw}+\frac{\sin\theta}{\text{Fr}} \tag{12}$$

$$\frac{\partial u}{\partial t}=-\frac{1}{R}\frac{\partial p}{\partial \xi}+\text{Conu}+\frac{1}{\text{Re}}\text{Diffu}-\frac{\cos\theta}{\text{Fr}} \tag{13}$$

where

$$\text{Conw}=-\frac{1}{R}\frac{\partial(uw)}{\partial \xi}-\frac{uw}{\xi R+R_2}-\frac{\partial w^2}{\partial z}+\frac{1}{R}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)\frac{\partial w^2}{\partial \xi} \tag{14}$$

$$\text{Diffw}=\frac{\partial^2 w}{\partial z^2}-\frac{2}{R}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)\frac{\partial^2 w}{\partial z\,\xi}+\frac{1}{R^2}\left(1+\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)^2\right)\frac{\partial^2 w}{\partial \xi^2}+\frac{1}{R}\left(\frac{1}{\xi R+R_2}+\frac{2}{R}\frac{\partial R}{\partial z}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)-\left(\xi\frac{\partial^2 R}{\partial z^2}+\frac{\partial^2 R_2}{\partial z^2}\right)\right)\frac{\partial w}{\partial \xi} \tag{15}$$

$$\text{Conu}=-\frac{1}{R}\frac{\partial u^2}{\partial \xi}-\frac{u^2}{\xi R+R_2}-\frac{\partial(uw)}{\partial z}+\frac{1}{R}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)\frac{\partial(uw)}{\partial \xi} \tag{16}$$

$$\text{Diffu}=\frac{\partial^2 u}{\partial z^2}-\frac{2}{R}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)\frac{\partial^2 u}{\partial z\,\xi}+\frac{1}{R^2}\left(1+\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)^2\right)\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{R}\left(\frac{1}{\xi R+R_2}+\frac{2}{R}\frac{\partial R}{\partial z}\left(\xi\frac{\partial R}{\partial z}+\frac{\partial R_2}{\partial z}\right)-\left(\xi\frac{\partial^2 R}{\partial z^2}+\frac{\partial^2 R_2}{\partial z^2}\right)\right)\frac{\partial u}{\partial \xi}-\frac{u}{(\xi R+R_2)^2} \tag{17}$$

Likewise, the initial and boundary conditions (6) – (9) are also transformed accordingly using equation (10), and $\xi\in[0,1]$.

$$w(\xi,z,t)=U_{\max}\left(1-\xi^2\right),u(\xi,z,t)=0 \text{ for } z=0 \tag{18}$$

$$w(\xi,z,t)=0=u(\xi,z,t) \text{ on } \xi=R_1(z) \text{ and } \xi=R_2(z), z_3\le z\le z_{\max} \tag{19}$$

$$\frac{\partial w(\xi,z,t)}{\partial \xi}=0, \text{ on } \xi=0,\ 0\le z\le z_3 \tag{20}$$

$$w(\xi,z,0)=0,\ u(\xi,z,0)=0,\ p(\xi,z,0)=0 \text{ for } z>0 \tag{21}$$

2.3 Method of Solution and Discretization Procedure

The above unsteady governing equations (11) - (13) are discretized using Marker and Cell (MAC) method [21]. The pressure and velocities are computed at different locations as shown in Figure 2. By defining $\xi=j\Delta\xi$, $z=i\Delta z$, $t=n\Delta t$ and $p(\xi,z,t)=p(j\Delta\xi,i\Delta z,n\Delta t)=p_{i,j}^n$, where n refers to the time t, Δt is the increment of time,while Δz, $\Delta\xi$ are the length and width of the *(i,j)* cell of the control volume respectively.

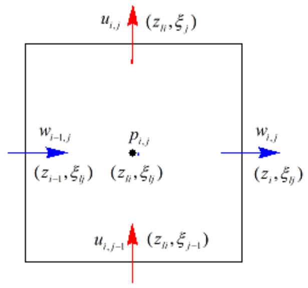

Figure 2. MAC Cell Method

4.2. Discretization of Governing Equations

The discretization of the continuity and momentum equations are performed at the (i,j)th cell to obtain:

$$(\xi_{lj}R_{li}^n+R_{2li}^n)\left(\frac{w_{i,j}^n-w_{i-1,j}^n}{\Delta z}\right)-\frac{\xi_{lj}R_{li}^n+R_{2li}^n}{R_{li}^n}\left(\xi_{lj}\left(\frac{\partial R}{\partial z}\right)_{li}^n+\left(\frac{\partial R_2}{\partial z}\right)_{li}^n\right)\left(\frac{w_{at}-w_{ab}}{\Delta\xi}\right)+\frac{(\xi_j R_i^n+R_{2i}^n)u_{i,j}^n-(\xi_{j-1}R_i^n+R_{2i}^n)u_{i,j-1}^n}{R_i^n\Delta\xi}=0 \tag{22}$$

where

$$w_{at}=\frac{w_{i,j}^n+w_{i-1,j}^n+w_{i-1,j+1}^n+w_{i,j+1}^n}{4} \tag{23}$$

$$w_{ab}=\frac{w_{i,j}^n+w_{i-1,j}^n+w_{i-1,j-1}^n+w_{i-1,j-1}^n}{4} \tag{24}$$

$$\xi_{lj}=\xi_j-\frac{\Delta\xi}{2},\quad R^n=R(z_{lj}),\ z_{lj}=z_i-\frac{\Delta z}{2}$$

and

$$\frac{w_{i,j}^{n+1}-w_{i,j}^n}{\Delta t}=\left(\frac{p_{i,j}^n-p_{i+1,j}^n}{\Delta z}\right)+\frac{1}{R_i^n}\left(\xi_{lj}\left(\frac{\partial R}{\partial z}\right)_i^n+\left(\frac{\partial R_2}{\partial z}\right)_i^n\right)\left(\frac{p_t-p_b}{\Delta\xi}\right)+wcd_{i,j}^n \tag{25}$$

$$wcd_{i,j}^n=\text{Conw}_{i,j}^n+\frac{1}{\text{Re}}\left(\text{Diffw}_{i,j}^n\right)+\frac{\sin\theta}{Fr} \tag{26}$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^n}{\Delta t}=-\frac{1}{R_{lj}^n}\left(\frac{p_{i,j}^n-p_{i,j+1}^n}{\Delta\xi}\right)+ucd_{i,j}^n \tag{27}$$

where $$ucd_{i,j}^n=\text{Conu}_{i,j}^n+\frac{1}{\text{Re}}\left(\text{Diffu}_{i,j}^n\right)-\frac{\cos\theta}{Fr} \tag{28}$$

where $\text{Conu}_{i,j}^n$ and $\text{Diffu}_{i,j}^n$ are convective and diffusive terms of the *u*- momentum equation at n^{th} time level at the (i,j)th cell. The terms are differences in the similar terms manner as in the *w*- equation of momentum. The complete numerical procedure already discussed in Alnussairy *et al* [15]. By using the result of *w*-velocity, volumetric flow rate (Q) and resistance to flow (λ) are described as:

$$Q_i^n=2\pi(R_i^n)^2\int_0^1\xi_j(w_{i,j})d\xi_j \tag{29}$$

$$\lambda=\frac{|\Delta p|}{Q_i^n} \tag{30}$$

3. Discussions

The numerical and simulations are performed using μ= 1.79×10^{-5} pa.s, ρ = 1.225 kg.m^{-3}, g= 9.8 m.s^{-2}, R_0 = 0.01085 m, $r_1^{'}$ = 0.0065, (radius of left bronchi), z_1=0. 092m, $\beta = 30^o$ (angle of branching between left branch and main branch) [5, 7], Δz = 0.1 and $\Delta\xi$ = 0.025.the Reynolds numbers of rest and normal conditional breathing is taken in the range of 800 to 2000 and also tracheal length z_{max} = 0.16 m in nondimensional. The solutions are generated by using a staggered grid of size 1600×40 at constant Fr = 0.27, Re = 1200 and U_0 = 85×10^{-2} m.s^{-1}. The pressure-based a finite-difference approximation is used to solve the unsteady governing PDE equations of motion. The results are found after the steady state is achieved in the simulation when the dimensionless t = 80. The pressure is computed to determine the velocity after solving of the momentum equations.

The velocity field is plotted for every slice to generate a complete description of the flow patterns in the trachea during resting and normal breathing with varying *Re*. The accuracy of the proposed method is validated with existing experimental data and numerical studies (Schroter and Sudlow [6]); Calay *et al.*[7]). Figure 3 compares the variation of axial velocity dependent radial position obtained by the present model with other findings for Re = 1570 and θ = 0^o. The *w*-velocity at slice A (z = 4) revealed a parabolic pattern. In addition, the laminar flow exhibited a maximum velocity in the central region and decreased to 0 close to the walls (Schroter and Sudlow,[6]). Conversely, the *w*-velocity at slice B (z = 12) is highly skewed toward the inner wall and lowered in the outer wall of the single bifurcation. This is in a good agreement with the findings of Calay *et al*.[7] on *w*-velocity for different axial positions in main and single bifurcated trachea.

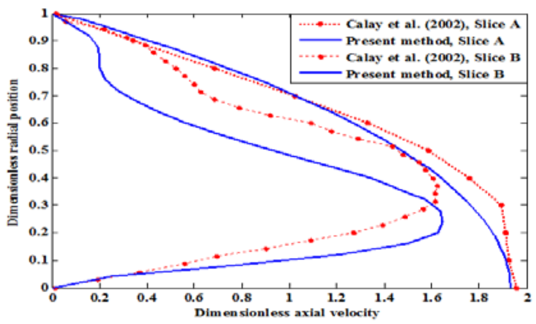

Figure 3. Comparison of the axial velocity with others for Re = 1570 when θ = 0^o

It calculated value of axial velocity in the horizontal straight trachea (θ = 0^o) for Re = 700 and slice B is compared with the experimental results of Schroter and Sudlow [6] as shown in Figure 4. The *w*-velocity near the outer wall is very slow. Thus, the velocity distribution is low in the outer and high in the inner wall.

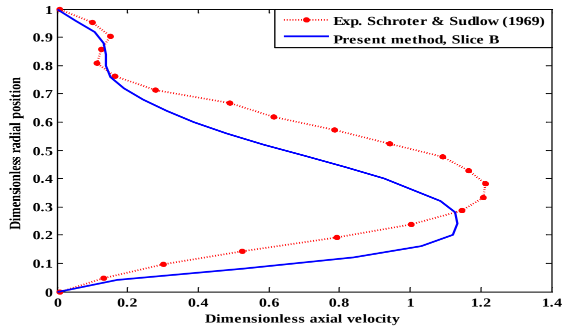

Figure 4 Comparison of the *w*-velocity in a bifurcated trachea position (slice B) with Re = 700 and θ = 0

Figure 5 presents the axial velocity dependent radial position at different axial locations for Re = 1200 and θ = 0^o. At z = 4 (slice A) the *w*-velocity revealed the parabolic shape inside the parent due to the prescribed boundary condition. However, the velocity distributions after the flow gets divided inside the daughter tube at other axial locations. The velocity at location z = 12 (slice B) near the outer wall is very slow, thus the velocity distribution is low in the outer and high in the inner wall. At z = 13.5 the velocity gradient is decreased very rapidly near the inner wall and it started rising in the outer wall to become M-shape. A typical M-shape, velocity is revealed at z = 16 (slice C). Furthermore, the velocity near the outer wall is high and reversed. This indicates the appearance of backflow separation near the outer wall for all values of *Re*.

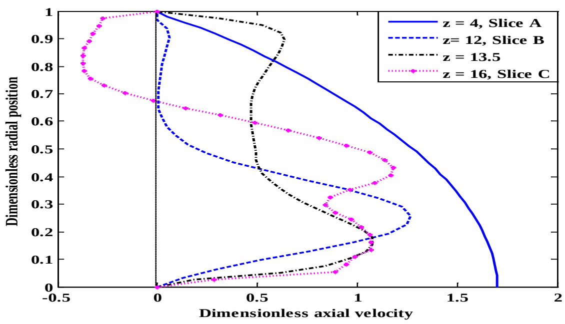

Figure 5. Variation of *w*-velocity for different position of z with Re = 1200 and θ = 0^o.

Figure 6 shows the *w*-velocity for different inclination angle position with fixed Re = 1200, Fr = 0.27 and z = 12 (slice B). An increase in the angle of inclination (from 0^o to 45^o) is found to enhance the airflow velocity. These results are in an agreement with the observation of Vliet *et al.* [3] & Ragavan *et al*. [4].

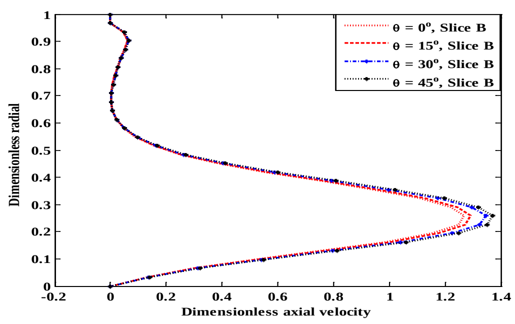

Figure 6. Variation of *w*-velocity for variance θ with Re = 1200, Fr = 0.27 and axial position (slice B)

In the human anatomy, pressure loss (wall) in the bifurcation airways plays a vital role in the process of respiration. The change of the pressure loss in human lung airways and alveoli is the driving force in the respiratory system. Thus, it is necessary to understand the effect of inclination angle and boundary condition on the pressure loss.
Figures 7 demonstrate the influence of the slope angle of the human trachea on the pressure loss ($\Delta p = p - p_0$), where p_0 signifies the inlet pressure. An increase in the slope angle situation is observed to enhance the pressure loss. It is because at a higher angle of inclination the velocity of airflow into the lungs during inhalation is increased. So, the increasing slope angle of the trachea increased the velocity of airflow through the main and the bifurcated trachea to the lungs. This in turn generated a greater negative pressure in the lungs. It supported the principle of Bernoulli's, where the pressure exerted by the gas is inversely related to the speed of the gas flow [22].

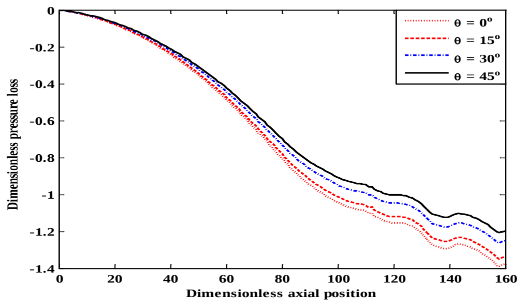

Figure 7 Variation of pressure loss with axial position for different θ with Re = 1200 and Fr = 0.27

Figure 8 presents the axial position dependent changes in the volumetric flow rate (Q) for different θ at Re = 1200 and Fr = 0.27. For higher slope position the value of Q inside a straight trachea is enhanced. Conversely, the flow rate through main bronchi is reduced because of the narrowing of the airway branch diameter. Moreover, the influence of slope angle (θ = 15^o) is found to be insignificant.

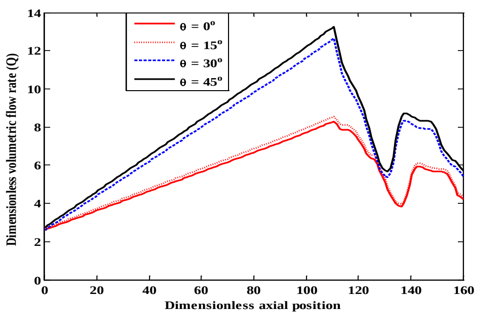

Figure 8 Variation of the volumetric flow rate with axial position for variance θ with Re =1200 and Fr = 0.27

According to Chovancova and Elener [23] flow of resistance in the airways depends on whether the flow is turbulent or laminar on the dimensions of the airway and on the gas viscosity. Therefore, the resistance to flow in a trachea at different θ is shown in Figure 9. It revealed an increase when going down the first bifurcated trachea. This implies a reduction in the airway branch diameter (cross-section area). It is clear that the flow resistance, reduced with increasing of slope position of θ.

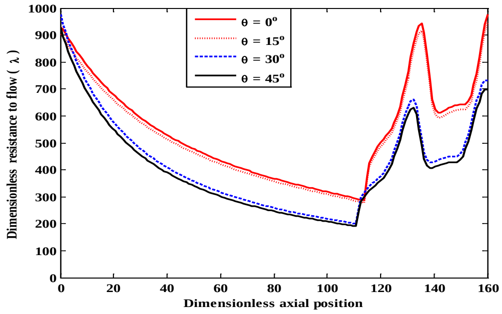

Figure 9 Difference of the resistance to flow with axial position for variance θ with Re = 1200 and Fr = 0.27

The understanding of airflow patterns through human trachea are significant to study the aerosolized medication delivery processes and localized diagnostics diseases of the lung. Figure 10 - 11 display axial position dependent streamlines behaviour of airflow for different values of θ. For θ = 0^o (Figures 10, 11) it is found that there are regions in the bifurcated trachea where the flow recirculation occurred irrespective of the *Re*. Furthermore, this recirculation is increased in the outer wall at the higher inclination angle of 30^o (Figure 11).

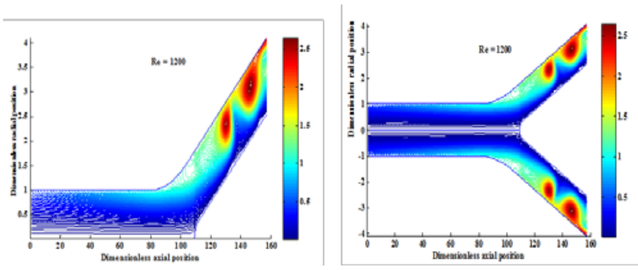

Figure10 Streamline of airflow pattern through main and single bifurcated trachea for Re=1200 at θ = 0^o

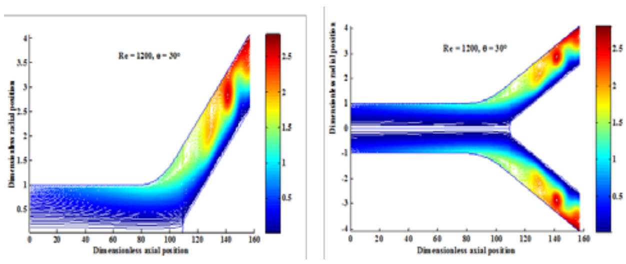

Figure 11 Streamline of airflow pattern through main and single bifurcated trachea for Re= 1200 at θ = 30^o

4. Conclusion

A symmetric mathematical model is developed and simulated to determine the effect of slope angle of the airflow through the trachea and main bronchi. The numerical model is simulated by using a MAC method with staggered grids. Increasing the slope angle is found to increase axial velocity, pressure loss and volumetric flow rate through the main trachea. The resistance to flow (pressure drop) is reduced with the increase of slope angle. The recirculation regions are increased in the outer wall with the higher slope angle. The sleeping in a horizontal situation leads to a negative influence for many patients. Thus, the slope angle situation between 30^o- 45^o is demonstrated to induce a better influence that may be helpful to the patients with chronic obstructive pulmonary disease and other respiratory diseases.

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Figure 11 Streamline of airflow pattern through main and single bifurcated trachea for Re= 1200 at $\theta = 30^o$

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Bifurcation in discrete prey-predator model

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Abstract: The dynamics of discrete-time prey-predator model are studied and investigated. The model has four fixed points. The origin fixed point is always exists while the others are exist under some conditions. The conditions that required achieving local stability of all fixed points are also set. The results indicate that the model has a flib bifurcation which found by varying the prey intrinsic growth parameter via pray and predator populations, respectively. Finally, numerical simulations not only illustrate our results, but also exhibit the complex dynamic behavior and chaotic.

Keywords: Discrete model, bifurcation theory, Competition.

1-Introduction:

Competition is an interaction between organisms or species in which both species are harmed. Competition may be for territory which is directly related to food resources. Some interesting phenomena have been found from the study of practical competition models. Hsu et al. [1] concerned with the growth of two predator species competing exploitatively for the same prey population. The predators feed on the prey with a saturating functional response to their prey density. The existence of species in the real world is not a lone so that the interaction, mutualism and competitive mechanisms are taken place. For that researchers have been investigated extensively in the recent years. They formed their models by using a set of differential equations [3,4,5]. Many authors have been carried out studying the chaotic dynamics that occur in multispecies continues time as well as discrete time prey-predator models [6,7,8]. In [9,10,11,12,13] authors have been given a modification of the system using nonlinear difference equations or partial differential equations .

Another example of competition is in Holt et al. [2]. They focused on the competition between two or more victim species that share a natural enemy. They also reviewed empirical examples of apparent competition in phytophagous insect hosts attacked by polyphagous parasitoids and they developed models of apparent competition in host-parasitoid systems. They found that the apparent competition is particularly likely in insect assemblages because parasitoids can limit their hosts to levels at which resource competition is unimportant.

This paper is organized as follows: in Section 2, the discrete prey–predator model is formulated and investigated, and then the conditions of existence and local stability of its fixed points are derived. In Section 3, we discussed that the model undergoes flip bifurcation in the interior ${R^2}_+$, by varying some values of parameters. Also, the numerical simulations are done to confirm the analytic results, such as the local stability as well as the bifurcation diagrams, phase portraits. Finally, in section 4 the conclusions are drawn.

2-The model and the analysis of its fixed points:

Consider the following discrete prey-predator model

$$\begin{cases} x_{t+1} = ax_t\left(1 - \frac{x_t}{1+y_t}\right) \\ y_{t+1} = cy_t\left(1 - \frac{y_t}{1+x_t}\right) \end{cases} \quad (1)$$

This model describes the interactions between two populations with the initial conditions x(0)>0 , y(0)>0 , where the x(t) and y(t) denote the number of prey and the number pf predator at time t, respectively. The parameters a and c are the growth rate of the two species, respectively. The possible fixed points are obtained by solving the following algebraic equations:

$$\begin{cases} x = ax\left(1 - \frac{x}{1+y}\right) \\ y = cy\left(1 - \frac{y}{1+x}\right) \end{cases}$$

With simple computation we get the following fixed points:

1) $e_1 = (0,0)$ is the origin fixed point which is always exists.

2) $e_2 = (r_1, 0)$, where $r_1 = \frac{a-1}{a}$, is the first axial fixed point which means the prey population exist with absence of predator one.

3) $e_3 = (0, r_2)$, where $r_2 = \frac{c-1}{c}$, is the second axial fixed point which means the predator population exist with absence of prey one.

4) $e_4 = (x^*, y^*) = (\frac{(1-a)(2c-1)}{1-(a+c)}, \frac{(1-c)(2a-1)}{1-(a+c)})$ is the unique positive fixed point which exist if and only if $a, c > 1$.

For studying the stability of each fixed point we shall obtain the variation matrix and its characteristic equation. In general with (x, y) is a fixed point of model (1), the Jacobian matrix at (x, y) can be written as;

$$J(x, y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$$

Where

$$j_{11} = a - \frac{2ax}{1+y}$$

$$j_{12} = \frac{ax^2}{(1+y)^2}$$

$$j_{21} = \frac{cy^2}{(1+x)^2}$$

$$j_{22} = c - \frac{2cy}{1+x}$$

and characteristic equation of $J((x, y))$ is:

$$F(\lambda) = \lambda^2 + P\lambda + Q \quad (2)$$

Where $P = c + a - \left(\frac{2cy}{1+x} + \frac{2ax}{1+y}\right)$ and

$$Q = \left(-\frac{2ax}{y+1} + a\right)\left(-\frac{2cy}{x+1} + c\right) - \frac{acx^2y^2}{(x+1)^2(y+1)^2}$$

Hence the system (1) is a dissipative system if $\left|\left(-\frac{2ax}{y+1} + a\right)\left(-\frac{2cy}{x+1} + c\right) - \frac{acx^2y^2}{(x+1)^2(y+1)^2}\right| < 1$ [12].

Let λ_1 and λ_2 be the two roots of equation(2), which are called the eigenvalues of the Jacobian matrix at any point. We recall some definitions of topological types for a fixed point. A fixed point is called a sink point if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink point is locally asymptotically stable. A fixed point is called a source point if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source point is locally unstable. A fixed point is called a saddle point if $|\lambda_1| > 1$ and $|\lambda_2| < 1$(or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). And a fixed point is called non-hyperbolic point if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$ [12]. The next propositions give the behavior dynamics of the fixed point e_1 as well as e_2 and e_3.

Proposition 2.1: The origin fixed point e_1 is:

a) Sink point if $a < 1$ and $c < 1$;
b) Source point if $a > 1$ and $c > 1$;
c) Non-hyperbolic point if $a = 1$ or $c = 1$;
d) Saddle point otherwise.

Proof: It is clear that the Jacobian matrix at e_1 is given as follows:

$$J_{e_1} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

Obviously, the eigenvalues of the J_{e_1} are $\lambda_1 = a$ and $\lambda_2 = c$, therefore all results can be obtained.

Proposition 2.2: For the fixed points e_2 and e_3 we have:

1- For the prey axial fixed point e_2 is:
a) Sink point if $1 < a < 3$ and $c < 1$;
b) Source point if $a > 3$ and $c > 1$;
c) Non-hyperbolic point if either $a = 1$or 3 or $c = 1$;
d) Saddle point otherwise.

2- For the predator fixed points there exist at least four different topological types these are:
a) Sink point if $a < 1$ and $1 < c < 3$;
b) Source point if $a > 1$ and $c > 3$;
c) Non-hyperbolic point if $a = 1$ either $= 1 or 3$;
d) Saddle point otherwise.

Proof: It is clear that the Jacobian matrices at $e_2$and e_3 are given by:

$$J_{e_2} = \begin{pmatrix} a-2ar_1 & ar_1{}^2 \\ 0 & c \end{pmatrix}$$

$$J_{e_3} = \begin{pmatrix} a & 0 \\ cr_1{}^2 & c-2cr_2 \end{pmatrix}$$

Hence, the eigenvalues of the J_{e_2} are $\lambda_1 = 2 - a$ and $\lambda_2 = c$ while the eigenvalues of the J_{e_3} are $\lambda_1 = a$ and $\lambda_2 = 2 - c$ therefore all results can be obtained, respectivaly.

Before studying the behavior of the unique positive fixed point e_4 , we need the following Lemma which appeared in[13]

Lemma 2.3 : Let $F(\lambda) = \lambda^2 + P\,\lambda + Q$. Suppose that $F(1) > 0, \lambda_1$ and λ_2 are the two roots of $F(\lambda) = 0$. Then

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;

(ii)$|\lambda_1| < 1$ and$|\lambda_2| > 1$ (or$|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$and $Q > 1$;

(iv) $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$.

Proof: see [13].

In order to discuss the dynamics behavior of the positive fixed point e_4, we need the Jacobian matrix at e_4 which is given by

$$J(x, y) = \begin{pmatrix} 2-a-\lambda & \frac{(a-1)^2}{a} \\ \frac{(c-1)^2}{c} & 2-c-\lambda \end{pmatrix}$$

Where P and Q in equation (2) are

$$P = a - 4 + c \text{ and}$$

$$Q = (a-2)(c-2) - \frac{(a-1)^2(c-1)^2}{ac}$$

Now, the next proposition gives the dynamics of the positive fixed point.

Proposition 2.4: The unique positive fixed pointe_4 is:

1- Sink point if and only if the $a \in (A,\infty) \cap I \cap [(0,B_2) \cup (B_1,\infty)]$

2- Source point if and only if the $a \in (A,\infty) \cap I \cap (B_2,B_1)$

3- Saddle point if $a \in (A,\infty) \cap [(0,\min\{b_1,b_2\}) \cup (\max\{b_1,b_2\},\infty)]$.

4- Non-hyperbolic point if $a \in (A_1,\infty)$ and either $a \neq 4-c \; or \; a \neq 2-c$;

Where

$A = \frac{2-c+\sqrt{(c-2)^2+4(c-1)}}{2}$

,$B_1 = \frac{2-c+\sqrt{(c-2)^2-4(c-1)^2}}{2}$,

$$B_2 = \frac{2-c-\sqrt{(c-2)^2-4(c-1)^2}}{2}$$

$I = (\min\{b_1,b_2\},\max\{b_1,b_2\})$,

$b_1 = \frac{-k_1+\sqrt{{k_1}^2-4k_2}}{2}$ and $b_2 = \frac{-k_1-\sqrt{{k_1}^2-4k_2}}{2}$ while $k_1 = \frac{c^2-5c-2}{c+1}$ and $k_2 = \frac{(c-1)^2}{c+1}$.

Proof: We will apply Lemma 2.3. Therefore:

$F(1) = 1+P+Q = 1+a-4+c+(a-2)(c-2)-\frac{(a-1)^2(c-1)^2}{ac} > 0$

That implies $a^2+(c-2)a-(c-1) > 0$. Thus $F(1) > 0$ if and only if $a \in (A_1,\infty)$.

Now, we have to show that $F(-1) > 0$ and $Q < 1$. So that, we have the following steps:

$F(-1) = 1-a+4-c+(a-2)(c-2)-\frac{(a-1)^2(c-1)^2}{ac} > 0$

That implies $a^2+\frac{(c^2-5c-2)}{c+1}a+\frac{(c-1)^2}{c+1} < 0$. Therefore $F(-1) > 0$ if and only if when

$a \in I$

It is clear that $Q = (a-2)(c-2)-\frac{(a-1)^2(c-1)^2}{ac} < 1$ if and only if $a^2-(2-c)a+(c-1)^2 > 0$. therefore $Q < 1$ if $a \in (\infty,B_2) \cup (B_1,\infty)$

According to the Lemma 2.3(1), $e_4$is sink when

$a \in (A_1,\infty) \cap I \cap [(0,B_2) \cup (B_1,\infty)]$

The proof of the other cases can be easily obtained.

3-Numerical simulation:

To provide some numerical evidence for the qualitative dynamic behavior of the model (1),so that at different set of values the local behavior of the all fixed points are investigated numerically. For the fixed point e_1 we choose the value of $a = 0.7$ and $c = 0.8$ as well as we choose the values $a = 1.7$ and $c = 0.8$ and $a = 0.7$ and $c = 1.8$ for the fixed points e_2 and e_3, respectively . Figures 1, 2, and 3 indicate the stability of e_1,e_2 and e_3 with the initail value (0.6,0.5). For the positive fixed point the values of $a = 1.8$ and $c = 1.2$ are chosen that satisfy the condition 1 in proposition 2.4. Figure 4 shows the local stability of the e_4=(0.55,0.26) with initial value (0.6,0.5).

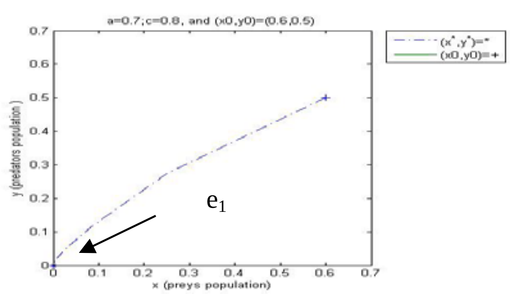


Figure 1: This figure shows the stability of e_1 according to the proposition 2.1.

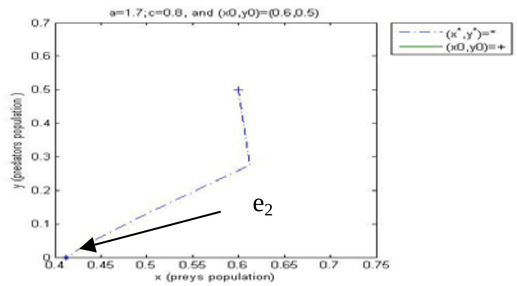


Figure 2:The stability of e_2 under the conditions of the proposition 2.2

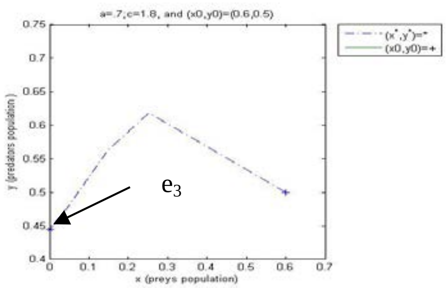


Figure 3: This figure shows the stability of e_3 according to the proposition 2.2

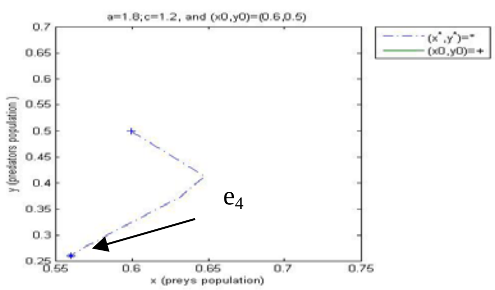


Figure 4: The stability of the positive fixed point e_4 according to the proposition 2.4

In different point of view, we study the phase portrait of the model (1) when we change only the parameter a via prey population and fix the others. To study the behavior of the model (1) when the parameter varied in the interval [0.75,3.95] one can consider the initial condition (0.6, 0.6) which is varied in the basin of attraction of positive fixed point e_4. When the control parameter varies, the stability of a periodic solution may be lost through various types of bifurcations and it gives the stable, period-2, period-4,perod-8,period-16,period-32 then chaotic

Now, without loss of generality we fix the parameters$c = 1.2$, and we assume that a is varied inside the interval[0.75,3.95]. The phase portraits are considered in the Figures 5,6,7,8,and 9:

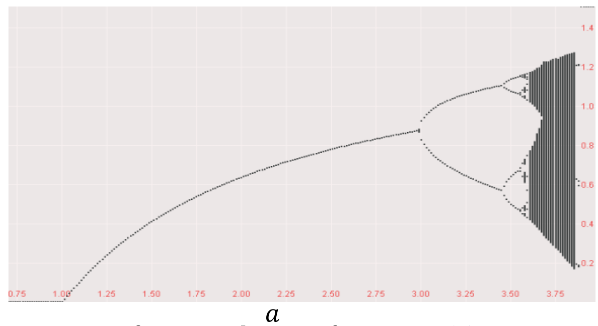
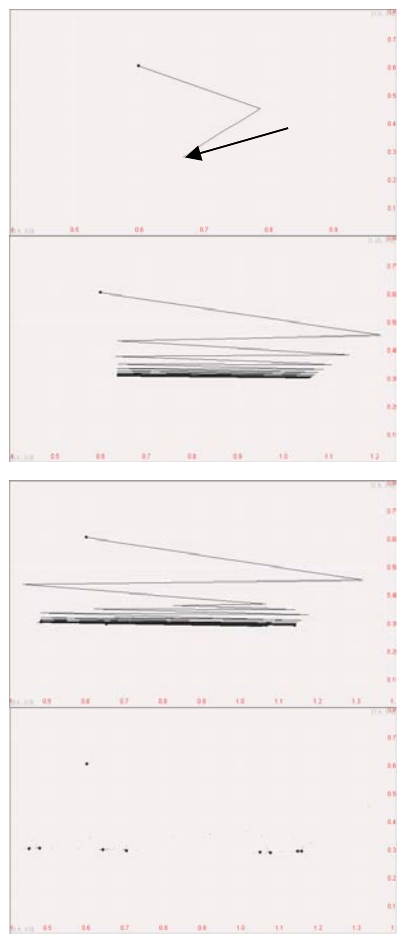


Figure5: Bifurcation diagram for system (1) versus a via prey population.



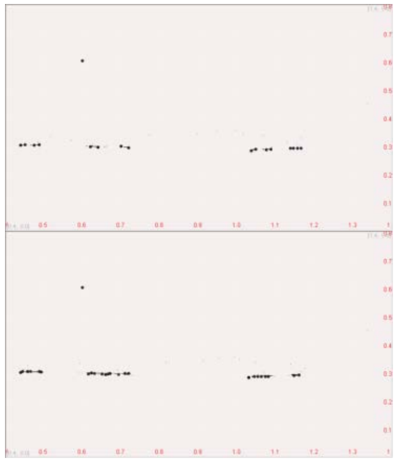


Figure6: These phase diagrams when $a = 2, 3.24, 3.5001, 3.544, 3.556, 3.5587$, respectively.

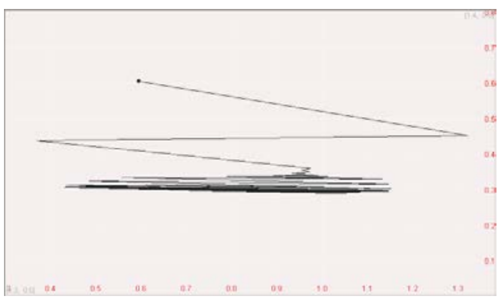
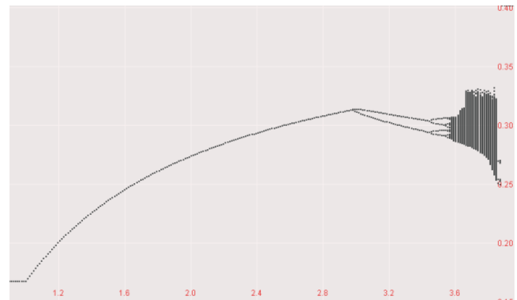


Figure7 : These phase diagrams gives the chaotic when $a = 3.564$.

The second numerical case starts when we will study the phase portrait of the model (1)as only the parameter a via predator population and fix the others. To study the behavior of the model (1) when the parameter varied in the interval [0.9,3.95] one can consider the initial condition (0.6, 0.6) situated in the basin of attraction of fixed point e_4. When the control parameter varies, the stability of a periodic solution may be lost through various types of bifurcations and it gives the stable, period-2, period-4 then chaotic.

Now, without loss of generality we fix the parameters $c = 1.2$, and we assume that $a \in [0.9, 3.95]$. The phase portraits are considered in the following Figures:



a

Figure 8: Bifurcation diagram for system (1) versus a via predator population.

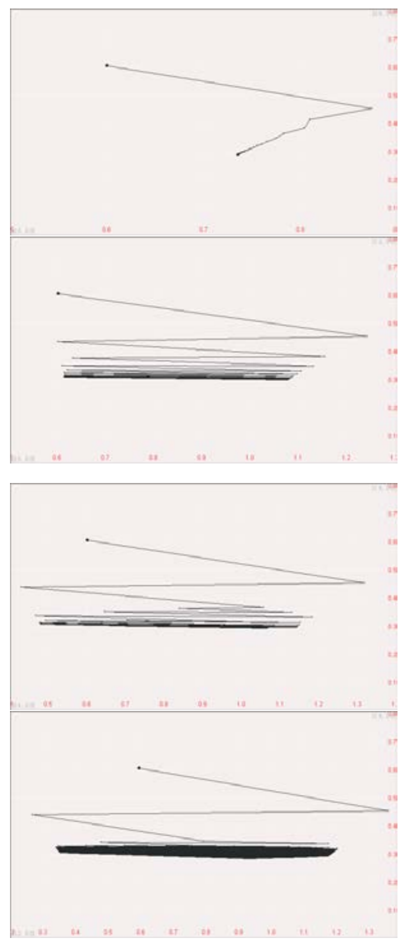


Figure 9: These phase diagrams give when $a = 2.33, 3.3, 3.507, 3.66$, respectively.

4-Conclusion:

In this paper, the local stability of all possible fixed points of a two dimensional discrete time prey-predator model has been studied and discussed. The chaotic dynamics and bifurcation of the model have been investigated. Basic properties of the model have been analyzed by means of phase portrait, and bifurcation diagrams. Under certain parametric conditions, the interior fixed point enters a flib bifurcation phenomenon. This could be very useful for the biologists as well as mathematicians who work with discrete-time prey–predator models.

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Study of a Predator-Prey Model with Modified Ratio-Dependent and Sokol-Howell Functional Response

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***Abstract*:** In this paper a predator-prey food chain model with modified ratio-dependent and Sokol-Howell functional response is proposed and discussed. The model is observed to be dissipative. The stability of the equilibrium points of the three species system is analyzed. The flow of the model is explored theoretically with two functional responses and numerically with three ones.

***Keywords*:** Sokol-Howell, modified ratio-dependent, stability analysis, functional response.

1. Introduction

As we all know that periodic and chaotic environmental models are eccentric in behavior. The permanence and extinction in predator-prey with ratio-dependent received attention by many ecological authors, see [1,3-6,11]. Jost and Arditi proves that prey and ratio-dependent systems can fit well with time arrangement created by each other [1]. Gakkhar and Naji in [6] studied the chaos in ratio-dependent model. Guin and Mandal [3] examined the flow of reaction-diffusion in ratio-dependent systems with intraspecific competition. Sokol-Howell functional response of the form $\frac{wx}{h+x^2}$ is studied by many ecologists; see [7-10]. In this paper, we modify the model of [8] by using the modified ratio-dependent Sokol-Howell functional response $\frac{wxy^2}{hy^2+x^2}$ in the place of the standard Sokol-Howell. The dynamics of the three-species predator-prey is studied (Stability analysis, Numerical exploration, results and conclusions), which shows the significance of the system beneath consideration.

2. The Mathematical Model

Consider the three species food chain model at time (t) consisting of the prey which is denoted by $x(t)$, the middle predator denoted by $y(t)$ and the top predator whose denoted by $z(t)$. The middle predator y preys on its only food x at the first level according to modified ratio-dependent Sokol-Howell functional response, while the top predator z preys on y at the second level according to the standard Sokol-Howell. The dynamics of the model can be represented by:

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - \frac{w_1 x y^2}{h_1 y^2 + x^2} = G_1(x,y,z),$$
$$\frac{dy}{dt} = \frac{w_2 x y^2}{h_2 y^2 + x^2} - d_1 y - \frac{w_3 y z}{h_3 + y^2} = G_2(x,y,z),$$
$$\frac{dz}{dt} = \frac{w_4 y z}{h_4 + y^2} - d_2 z = G_3(x,y,z). \qquad (1)$$

The functional response in system (1) is proposed by removing the prey x and put the ratio $\frac{x}{y}$ in Sokol-Howell response. The solution of the system (1) exists and is unique since all the functions G_i $(i=1,2,3)$ are Lipschitzian on $R^3_+ = \{(x,y,z) \in R^3 : x \geq 0, y \geq 0, z \geq 0\}$. Here the positive constants a_1, b_1, d_j $(j=1,2)$ h_k and w_k $(k=1,2,3,4)$ denote to: a_1 is the growth rate of the prey x, b_1 represents the intraspecific competition of prey x, w_k's are the maximum values attainable by each per capita rate, h_k's are the half-saturation constant, d_j's represent the death rate of the middle and the top predators.

<u>Note</u>: System (1) is observed to be dissipative, see [8].

3. Stability Analysis

In this section, the stability of the equilibrium points of model (1) is discussed. The points $E_0 = (0,0,0)$ and $E_1 = (\frac{a_1}{b_1},0,0)$ are always exist. The third equilibrium point given by $E_2 = (x_*, y_*, 0)$ exists where

$$x_* = \frac{1}{b_1}\left(a_1 - \frac{w_2^3}{h_1 w_2^2 + 4 d_2^2 h_2^2}\right) \quad \text{and} \quad y_* = \frac{w_2 x}{2 d_2 h_2} \qquad (2)$$

with the following condition provided that $0 < x < \frac{a_1}{b_1}$

$$x^2 (v_2 - 4 d_2^2 h_2) = 0. \qquad (3)$$

For the stability analysis of E_0, E_1 and E_2 see [8].

Now, the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ exists if there is appositive solution to the following equations in the $Int.R^3_+$.

$$g_1 = a_1 - b_1 x - \frac{w_1 y^2}{h_1 y^2 + x^2} = 0,$$

$$g_2 = \frac{w_2 xy}{h_2 y^2 + x^2} - d_1 - \frac{w_3 z}{h_3 + y^2} = 0,$$

$$g_3 = \frac{w_4 y}{h_4 + y^2} - d_2 = 0. \quad (4)$$

From the third equation of (4) we have

$$d_2 y^2 - w_4 y + d_2 h_4 = 0, \quad (5)$$

so that,

$$y^* = \frac{w_4 \pm \sqrt{w_4^2 - 4d_2^2 h_4}}{2d_2},$$

Hence, if the term $w_4^2 - 4d_2^2 h_4 < 0$, then there is no positive solution to Eq. (5) and if $w_4^2 - 4d_2^2 h_4 > 0$, then there are two positive solution to Eq. (5). The last case occurs if the following condition holds

$$w_4^2 - 4d_2^2 h_4 = 0. \quad (6)$$

Then, there is only one solution given by

$$y^* = \frac{w_4}{2d_2}. \quad (7)$$

From the first equation of (4)

$$b_1 x^{*3} - a_1 x^{*2} + b_1 h_1 x^* y^{*2} + y^{*2}(w_1 - a_1 h_1) = 0. \quad (8)$$

Equation (8) has one positive root depending on Descartes's rule if

$$w_1 < a_1 h_1. \quad (9)$$

Again, from the second equation of (4)

$$z^* = \frac{h_3 + y^{*2}}{w_3}\left[\frac{w_2 x^* y^*}{h_2 y^{*2} + x^{*2}} - d_1\right], \quad (10)$$

Now, in addition to condition (6) and (9) the positive point E_3 exists if the following condition holds

$$\frac{w_2 x^* y^*}{h_2 y^{*2} + x^{*2}} > d_1. \quad (11)$$

The varational matrix $V = (x, y, z)$ is computed for system (4) as:

$$V(x, y, z) = \lfloor m_{ij} \rfloor \; i, j = 1,2,3, \quad (12)$$

where

$$m_{11} = a_1 - 2b_1 x^* - \frac{w_1 y^{*2}(h_1 y^{*2} - x^{*2})}{(h_1 y^{*2} + x^{*2})^2}$$

$$m_{12} = -\frac{2w_1 x^{*3} y^*}{(h_1 y^{*2} + x^{*2})^2}$$

$$m_{13} = 0,$$

$$m_{21} = \frac{w_2 y^{*2}(h_2 y^{*2} - x^{*2})}{(h_1 y^{*2} + x^{*2})},$$

$$m_{22} = \frac{2w_2 h_2 x^* y^{*3}}{(h_1 y^{*2} + x^{*2})^2} - d_1 - \frac{w_3 z^*(h_3 - y^{*2})}{(h_3 + y^{*2})^2},$$

$$m_{23} = -\frac{w_3 y^*}{(h_3 + y^{*2})},$$

$$m_{31} = 0,$$

$$m_{32} = \frac{w_4 z^*(h_4 - y^{*2})}{(h_4 + y^{*2})^2},$$

$$m_{33} = \frac{w_4 y^*}{(h_4 + y^{*2})} - d_2.$$

The characteristic equation of the above matrix (12) can be written as:

$$\lambda^3 + H_1\lambda^2 + H_2\lambda + H_3 = 0,$$

where

$$H_1 = -(m_{11} + m_{22} + m_{33}),$$

$$= -\left(a_1 - 2b_1 x^* - \frac{w_1 y^{*2}(h_1 y^{*2} - x^{*2})}{P_1^2}\right) - \left(\frac{2w_2 x^{*3} y^*}{P_2^2} - d_1 - \frac{w_3 z^*(h_3 - y^{*2})}{Q_1^2}\right) - \left(\frac{w_4 y^*}{Q_2^2} - d_2\right),$$

where

$P_1 = (h_1 y^{*2} + x^{*2}),\ P_2 = (h_2 y^{*2} + x^{*2})$ and

$Q_1 = (h_3 + y^{*2}),\ Q_2 = (h_4 + y^{*2})$.

Similarly we write H_2, H_3 and $H_1 H_2 - H_3$ in the form of m_{ij}, where

$$H_2 = (m_{11}m_{22} - m_{12}m_{21}) + (m_{22}m_{33} - m_{23}m_{32}) + m_{11}m_{33}$$

$$H_2 = (m_{11}m_{22} - m_{12}m_{21}) + m_{22}m_{33} + m_{11}m_{33}.$$

Since $y^{*2} = h_4$ according to condition (6).

$$H_3 = m_{33}(m_{12}m_{21} - m_{11}m_{22}),$$

and

$$H_1 H_2 - H_3 = -(m_{11} + m_{22})[(m_{11}m_{22} - m_{12}m_{21}) + m_{22}m_{33} + m_{11}m_{33}] - m_{33}^2(m_{11} + m_{22})$$

Now, straightforward computations show that $H_1 > 0$, $H_3 > 0$, and $H_1 H_2 - H_3 > 0$ if and only if the next conditions are hold:

$$\frac{1}{2x^*}\left[a_1 - \frac{w_1 y^*(h_1 y^{*2} - x^{*2})}{P_1^2}\right] < b_1, \quad (13)$$

$$h_1,\, h_2 > \frac{x^{*2}}{y^{*2}} \tag{14}$$

$$\frac{2w_2 h_2 x^* y^{*3}}{(h_1 y^{*2} + x^{*2})^2} - \frac{w_3 z^* (h_3 - y^{*2})}{(h_3 + y^{*2})^2} < d_1 , \tag{15}$$

$$h_3 > y^{*2} , \tag{16}$$

$$\frac{w_4 y^*}{Q_2} < d_2 . \tag{17}$$

According to Routh-Hurwitz criterion, $E_3 = (x^*, y^*, z^*)$ is locally asymptotically stable in the $Int.R_+^3$ provided conditions (13-17) hold, see [2, 13].
Now, for the global asymptotic stability we didn't find a suitable Lyapunov function and we discuss the global dynamics numerically in the next section.

4. Numerical Exploration

The Runge-Kutta method of six order is used to solve the system (1) numerically, see [12]. There are two cases here to discuss. The first case of system (1) itself, and the second case we replacing the Sokol-Howell functional response by Leslie-Gower and we run the new system numerically so that to analyze the behavior of modified ratio-dependent functional response more.

4.1 Modified Ratio-Dependent with Sokol-Howell

For the following data set

$$a_1 = 0.20 ,\quad b_1 = 0.0007 ,\quad w_1 = 0.051,\quad w_2 = 0.27 ,$$
$$w_3 = 0.21,\ w_4 = 0.095 ,\ d_1 = 0.0033 ,\ d_2 = 0.005,$$
$$h_1 = h_2 = 0.22 ,\ h_3 = 1.0 . \tag{18}$$

The attractors for model (1) are plotted depending on the half-saturation constant h_4 of the top predator, since we discussed and other authors the effects of the growth rate, death rate and the intraspecific competition in many papers, see [3,8,9].

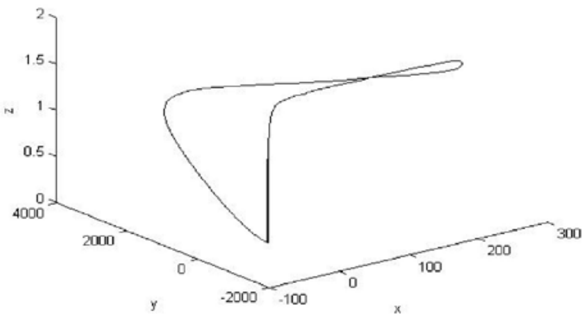

Figure 1. 3D of system (1) period 2 with data (18) and $h_4 = 1.20$ with fading in the top predator.

For h_4 with data (18), system (1) observed to be with period 2 and vanishing of the top predator as it shown in figure 1. Decreasing the value of h_4 from 0.9 to 0.5, then model (1) is periodic with period 1 as plotted in figure 2. Decreasing h_4 a little bit more for $h_4 = 0.4$, then system (1) food chain is stable as it shown in figure 3.

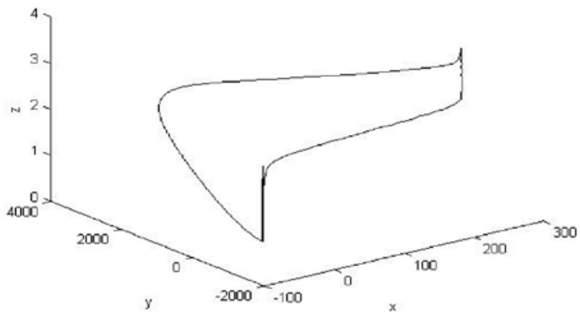

Figure 2. 3D of system (1) periodic with data (18) and $h_4 = 0.5$ with extinction in the top predator.

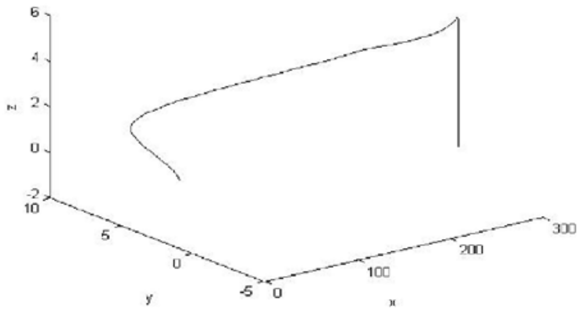

Figure 3. 3D of model (1) with data (18) stable for $h_4 = 0.4$

4.2 Modified Ratio-Dependent with Leslie-Gower

The food chain system (1) is modified numerically by putting the Leslie-Gower in the place of Sokol-Howell in third equation of system (1) and the top predator equation written as:

$$\frac{dz}{dt} = c_3 z^2 - \frac{w_4 z^2}{h_4 + y} , \tag{19}$$

and also we don't forget to replace y^2 in the denominator of last term of the middle predator by y and the last term change to $\frac{w_3 yz}{h_3 + y}$. The model in [8], we used the standard Sokol-Howell with Leslie-Gower and the model exhibits chaotic dynamics. Now, for the following data set

$$a_1 = 2.50 ,\quad b_1 = 0.5 ,\quad w_1 = 0.25 ,\quad w_2 = 7.5 ,$$
$$w_3 = 0.21,\ w_4 = 1.925 ,\ d_1 = 0.0042 ,\ c_3 = 0.005,$$
$$h_1 = h_2 = 20.0 ,\ h_3 = h_4 = 10.0 , \tag{20}$$

We run the Lesile-Gower with modified ratio-dependent for data (20) and our target to see the changes in the behavior of the system dynamics and also comparing our results in section **5** with the model in [8].

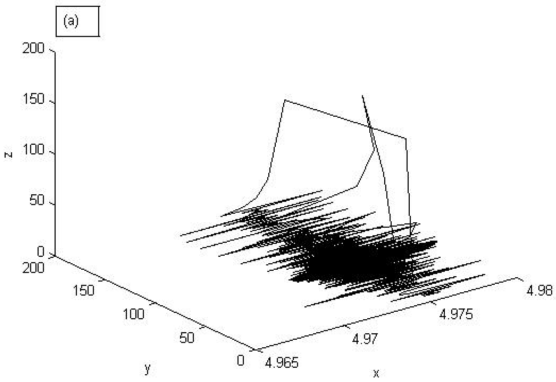

Figure 4(a). 3D of modified ratio-dependent and Leslie-Gower with data (20), stable with persistence of the prey x**,** middle predator y and the top predator z **.**

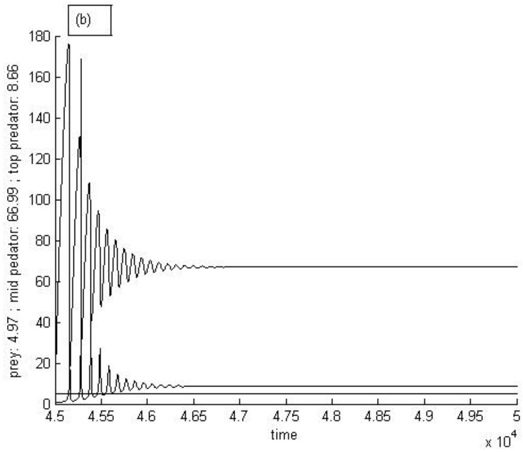

Figure 4(b). Time series of figure (5a).

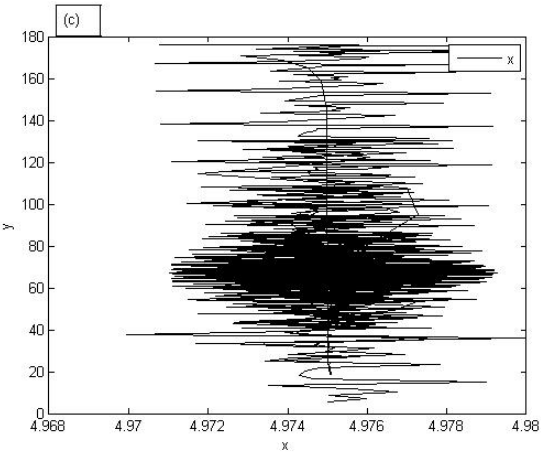

Figure 4(c). 2D xy-plane of figure (5a), stable of the prey and periodic turn to stable of the middle predator.

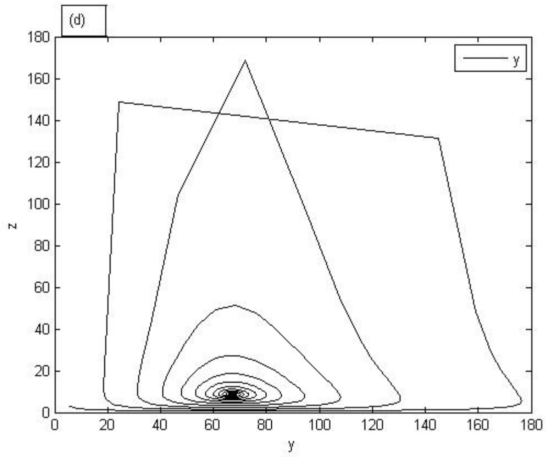

Figure 4(d). 2D yz-plane of figure (5a) periodic change to stable.

5. Results and Conclusions

The model (1) is investigated theoretically and figures of the attractors are blotted in **Figs. 1-3** with data (18) for the modified ratio-dependent with Sokol-Howell and in **Figs. 4(a-d)** with data (20) for modified ratio-dependent with Leslie-Gower. Now, for data (18) we depend on the control parameter the half-saturation level h_4 of the top predator while we depend in data (20) completely, and results after that are obtained:

1) For the value of $h_4 = 1.2$ system (1) shows the periodic as in Fig. 1, while decreasing the value of h_4 from 0.9-05 and 0.4 change the system to less periodic and then to stable as blotted in Fig. 2-3, so the saturation level h_4 is the control parameter of the food chain (1).
2) The permanence of the periodic of the system with fading of the top predator z , so that the model is not complicated as with standard Sokol-Howell functional response.
3) Changing the third term of the model from Sokol-Howell to Leslie-Gower with data (20) turn the system from periodic to stable with coexisting of all the species of the model and less density of the prey x as it plotted in Fig. 4.
4) The model in [8], we used the standard Sokol-Howell with Leslie-Gower and the model exhibits chaotic dynamics while system (1) described above is periodic with nearly the same data.
5) A three of functional responses are used here after putting the Leslie-Gower in the last equation of (1) and Holling type II in the place of Sokol-Howell in the second equation of system (1).

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Property (ao) AND TENSOR PRODUCT

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ABSTRACT: Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$ are a continuous linear operators and both have property (ao) then their tensor product has property (ao) if and only if the upper Weyl spectrum identity $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2)\cup\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ holds true. Perturbations by quasi-nilpotent operators are considered.

1. INTRODUCTION

We will postulate along this paper X is a Banach space and BL(X) refer to each a continuous linear operators on X. For $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$, let $\sigma(\mathcal{S}), \sigma_a(\mathcal{S})$ and iso $\sigma(\mathcal{S})$denote respectively the spectrum, the approximate point spectrum and isolated points of $\sigma(\mathcal{S})$. Let $\alpha(\mathcal{S})$ refer to the nullity of $\mathcal{S}$ defined by $\alpha(\mathcal{S}) = \dim\ker(\mathrm{S})$ and $\beta(\mathcal{S})$ refer to the deficiency of $\mathcal{S}$ defined by $\beta(\mathcal{S}) = \mathrm{codim}\,\mathrm{S}(\mathrm{X})$. If nullity of $\mathcal{S}$ is finite and rang of $\mathcal{S}$ ($\Re(\mathrm{S})$) is closed then $\mathcal{S}$ is called an upper semi-Fredholm operator and if deficiency of $\mathcal{S}$ is finite then $\mathcal{S}$ is a lower semi-Fredholm operator.

In the complete $\varphi_+(\mathrm{X})$ (resp. $\varphi_-(\mathrm{X})$) denote the set of all upper (resp. lower) semi-Freadholm operators on X. A continuous linear operator $\mathcal{S}$ is either upper or lower semi-Fredholm then $\mathcal{S}$ is semi-Fredholm (symbolizes $\varphi_+^-(\mathrm{X})$). While $\mathcal{S}$ is called a Freadholm operator (symbolizes $\varphi(\mathrm{X})$) if nullity and deficiency of $\mathcal{S}$ are finite. Now we can introduce the definition of an upper Weyl spectrum of $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \notin \varphi_+^-(\mathrm{X})\}$. $\mathrm{ind}(\mathcal{S})$ pointing to the index of $\mathcal{S}$ and defined as follows $\mathrm{ind}(\mathcal{S}) = \alpha(\mathcal{S}) - \beta(\mathcal{S})$. The ascent of $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$ is littlest non-negative integer $\mathcal{p} = \mathcal{p}(\mathcal{S})$ such that $\ker\mathcal{S}^{\mathcal{p}} = \ker\mathcal{S}^{\mathcal{p}+1}$, if there is not such integer then $\ker\mathcal{S}^{\mathcal{p}} \neq \ker\mathcal{S}^{\mathcal{p}+1}$ for each $\mathcal{p}$, then $\mathcal{p}(\mathcal{S})$ is infinite. And the descent of an operator $\mathcal{S}$ is littlest non-negative integer $\mathcal{q} = \mathcal{q}(\mathcal{S})$ such that $\mathcal{S}^{\mathcal{q}}(\mathrm{X}) = \mathcal{S}^{\mathcal{q}+1}(\mathrm{X})$, if there is not such integer $\mathcal{S}^{\mathcal{q}}(\mathrm{X}) \neq \mathcal{S}^{\mathcal{q}+1}(\mathrm{X})$ for each $\mathcal{q}$ then $\mathcal{q}(\mathcal{S})$ is infinite. According to [1], the ascent and the descent are equal if $\mathcal{p}(\mathcal{S})$ and $\mathcal{q}(\mathcal{S})$ are finite.

A continuous linear operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$ is Weyl if $\mathcal{S}$ is Fredholm of index zero, whilst is said to be Browder if $\mathcal{S} \in \varphi(\mathrm{X})$ and $\mathcal{p}(\mathcal{S}), \mathcal{q}(\mathcal{S})$ are finite. The Weyl, Browder and Browder approximate point spectrum define as follows

$$\sigma_W(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not Weyl}\},$$

$$\sigma_b(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not Browder}\},$$

$$\sigma_{ab}(\mathcal{S}) = \{\eta \in \sigma_a(\mathcal{S}): \eta \notin \varphi_+(\mathcal{A})\text{and } \mathcal{p}(\mathcal{S} - \eta) = \infty\}.$$

An operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$ is satisfies Weyl's Theorem if $\sigma(\mathcal{S})\backslash\sigma_W(\mathcal{S}) = \mathrm{E}^0(\mathcal{S})$ and satisfies Browder's Theorem if $\sigma(\mathcal{S})\backslash\sigma_W(\mathcal{S}) = \Pi^0(\mathcal{S})$ where $\mathrm{E}^0(\mathcal{S})$ is the eigenvalue of finite multiplicity and $\Pi^0(\mathcal{S})$ is poles of $\mathcal{S}$. We can say also a-Weyl's Theorem holds for $\mathcal{S}$ if $\sigma_a(\mathcal{S})\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \mathrm{E}_a^0(\mathcal{S})$ and a-Browder's Theorem holds for $\mathcal{S}$ if $\sigma_a(\mathcal{S})\backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \Pi_a^0(\mathcal{S})$ where $\mathrm{E}_a^0(\mathcal{S})$ an eigenvalue of $\mathcal{S}$ of finite multiplicity that isolated in approximate point spectrum of $\mathcal{S}$ and $\Pi_a^0(\mathcal{S})$ is left poles of $\mathcal{S}$ of finite rank.

And we continuous to narrate the theories, but before this we will impose n is non-negative integer and $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$ define $\mathcal{S}_{[n]}$ to be restriction of $\mathcal{S}$ to $\Re(\mathcal{S}^n)$ are seen as a map from $\Re(\mathcal{S}^n)$ into $\Re(\mathcal{S}^n)$, [special case $\mathcal{S}_{[0]} = \mathcal{S}$]. For some integer n, if the rang space $\Re(\mathcal{S}^n)$ is closed and $\mathcal{S}_{[n]}$ is an upper semi- Fredholm operator, then $\mathcal{S}$ is said to be upper semi B-Fredholm, while if the rang space $\Re(\mathcal{S}^n)$ is closed and $\mathcal{S}_{[n]}$ is a lower semi-Fredholm operator, then $\mathcal{S}$ is called lower semi B −Fredholm. The index of $\mathcal{S}$ is defined as the index of operator.

For $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$, is called B −Weyl if it a B −Fredholm operator of index zero, and so B −Weyl spectrum of $\mathcal{S}$ is defined by $\sigma_{Bw}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not B} - \text{Weyl}\}$. So we can say that an operator $\mathcal{S}$ achieves generalized Weyl's Theorem if $\sigma(\mathcal{S})\backslash\sigma_{Bw}(\mathcal{S}) = \mathrm{E}(\mathcal{S})$, and achieves generalized Browder's Theorem if $\sigma(\mathcal{S})\backslash\sigma_{Bw}(\mathrm{S}) = \Pi(\mathcal{S})$, where E(S) is an eigenvalue of $\mathcal{S}$ that are isolated in spectrum of $\mathcal{S}$ and $\Pi(\mathrm{S})$ is a poles of resolvent of S. The class of all upper semi B −Fredholm operators we will signal to him $\mathcal{SBF}_+(\mathrm{X})$ whereas $\mathcal{SBF}_+^-(\mathrm{X}) = \{\eta \in \mathcal{SBF}_+(\mathrm{X}): \mathrm{ind}(\mathcal{S}) \leq 0\}$, thus it will be defined the upper B −Weyl spectrum is $\sigma_{\mathcal{SBF}_+^-}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \notin \mathcal{SBF}_+^-(\mathrm{X})\}$. Hence after definition upper B −Weyl spectrum we call recall generalized a-Weyl's Theorem and generalized a-Browder's Theorem alternately, $,\sigma_a(\mathcal{S})\backslash \sigma_{\mathcal{SBF}_+^-}(\mathcal{S}) = \mathrm{E}_a(\mathcal{S})$ and $\sigma_a(\mathcal{S})\backslash\ \sigma_{\mathcal{SBF}_+^-}(\mathcal{S}) = \Pi_a(\mathcal{S})$, where $\mathrm{E}_a(\mathcal{S})$ is an eigenvalue of $\mathcal{S}$ that are isolated in approximate point spectrum of $\mathcal{S}$ and $\Pi_a(\mathcal{S})$ is a left poles of $\mathcal{S}$. Remain to mention the definition of Drazin spectrum and left Drazin invertible spectrum, if $\mathcal{S}$ has finite ascent and descent then $\mathcal{S}$ is called Drazin invertible, the Drazin spectrum $\sigma_D(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not a Drazin invertible}\}$. An operator $\mathcal{S}$ is called left Drazin invertible (in symbol LD(X)), if $\mathrm{LD}(\mathrm{X}) = \{\mathcal{S} \in \mathrm{BL}(\mathrm{X}): \mathcal{p}(\mathcal{S}) < \infty \text{ and } \Re(\mathcal{S}^{\mathcal{p}(\mathcal{S})+1}) \text{ is closed}\}$, and left Drazin invertible spectrum is defined by $\sigma_{LD}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \notin \mathrm{LD}(\mathrm{X})\}$.

Recall that a continuous linear operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$, has single valued extension property at a point $\eta_0 \in \mathbb{C}$ (Shortly $\mathcal{SVEP}$), if for every open disc $\mathcal{U}$ centered at η_0 then only analytic function f: $\mathcal{U} \longrightarrow \mathcal{A}$ satisfying $(\mathcal{S} - \eta)\mathrm{f}(\eta) = 0$ is the function $\mathrm{f} \equiv 0$. Evidently, $\mathcal{S}$ has $\mathcal{SVEP}$ at every isolated point of the spectrum, consequently, note that the single valued extension property plays an important role in Fredholm and spectral Theory.

We postulate that $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, the tenors product of two operators $\mathcal{S}_1$and $\mathcal{S}_2$ on $\mathrm{X}_1\otimes\mathrm{X}_2$ is the operator $\mathcal{S}_1\otimes\mathcal{S}_2$ defined by $(\mathcal{S}_1\otimes\mathcal{S}_2)\sum_i x_{1i}\otimes x_{2i} = \sum_i \mathcal{S}_1 x_{1i}\otimes\mathcal{S}_2 x_{2i}$ for all $\sum_i x_{1i}\otimes x_{2i} \in \mathrm{X}_1\otimes\mathrm{X}_2$. [6,8], if $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy Browder's Theorem then $\mathcal{S}_1\otimes\mathcal{S}_2$ satisfies Browder's Theorem if and only if the Weyl spectrum identity $\sigma_w(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma_w(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_w(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ holds, and if $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy a-Browder's Theorem then $\mathcal{S}_1\otimes\mathcal{S}_2$ satisfies a-Browder's Theorem if and only if the upper Weyl spectrum identity $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma_a(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma_a(\mathcal{S}_1)$ holds.

2. Property (ao) and tensor product

The most important findings of this paper is, if $\mathcal{S}_1 \in BL(X_1)$ and $\mathcal{S}_2 \in BL(X_2)$ have property (ao) then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao) if and only if the upper Weyl spectrum identity $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ holds, also study perturbation under a quasi-nilpotent operator for these royalty, this is part of the study, While the other is assume that $\mathcal{S}_1 \in BL(X_1)$ and $\mathcal{S}_2 \in BL(X_2)$ are polaroid and $\mathcal{S}_1^*$, $\mathcal{S}_2^*$ have $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (SZ), and study perturbation by commutator a quasi-nilpotent operator for property (SZ). The following lemmas help to reach the desired results: [1, Theorem 3.23], If $\mathcal{S} \in BL(X)$ has $\mathcal{SVEP}$ at $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+}(\mathcal{S})$ then $\eta \in \text{iso}\, \sigma_a(\mathcal{S})$ and $p(\mathcal{S} - \eta) < \infty$. From [4] and [11] we get the following results

i- $\sigma_x(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_x(\mathcal{S}_1)\sigma_x(\mathcal{S}_2)$, where $\sigma_x = \sigma$ or $\sigma_x = \sigma_a$,

ii $-$ $\sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+}(\mathcal{S}_1)\sigma_a(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+}(\mathcal{S}_2)\sigma_a(\mathcal{S}_1)$,

iii $-$ $\sigma_{\mathcal{SF}_-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_-}(\mathcal{S}_1)\sigma_\delta(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+}(\mathcal{S}_2)\sigma_\delta(\mathcal{S}_1)$.

and proposition 3 in [12], we obtain $\text{iso}\, \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \subset \text{iso}\, \sigma(\mathcal{S}_1)\, \text{iso}\, \sigma(\mathcal{S}_2)$.

Lemma 2.1 Let $\mathcal{S}_1$, $\mathcal{S}_2$ are a continuous linear operators in $BL(X_1)$ and $BL(X_2)$ respectively, then $0 \notin \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \backslash \sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

proof: We assume that $0 \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$ that is $\mathcal{S}_1 \otimes \mathcal{S}_2$ is not invertible and therefore $0 \in \text{iso}\, \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and from [1, Theorem 3.18], $\mathcal{S}_1 \otimes \mathcal{S}_2$ has $\mathcal{SVEP}$. And $0 \notin \sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2)$, so that $\mathcal{S}_1 \otimes \mathcal{S}_2$ has closed rang and $0 < \alpha(\mathcal{S}_1 \otimes \mathcal{S}_2) < \infty$. Since $\mathcal{S}_1 \otimes \mathcal{S}_2$ is surjective and has $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ is injective [1, corollary 2.24], consequently $\mathcal{S}_1$and $\mathcal{S}_2$ are injective if and only if $\mathcal{S}_1 \otimes \mathcal{S}_2$ is injective, we obtain $\alpha(\mathcal{S}_1) > 0$ or $\alpha(\mathcal{S}_2) > 0$. But $\alpha(\mathcal{S}_1 \otimes \mathcal{S}_2)$ is infinite, this leads to a discrepancy

Lemma 2.2 Let $\mathcal{S}_1 \in BL(X_1)$ and $\mathcal{S}_2 \in BL(X_2)$, then $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$

$$\subseteq \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2).$$

Proof: The inclusion $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1) \subseteq \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ verified because $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) \subseteq \sigma_{ab}(\mathcal{S})$ for all operator $\mathcal{S}$. Now we must prove that $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$, let $\eta \notin \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ as $\sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ implies that $\eta \neq 0$. Presume $\eta = \hbar\ell$ be any factorization of η, we obtain $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$ and therefor $\hbar \in \sigma(\mathcal{S}_1) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)$. Then $\hbar \in \varphi_+(\mathcal{S}_1)$, ind $(\mathcal{S}_1 - \hbar) \leq 0$, and $\ell \in \varphi_+(\mathcal{S}_2)$, ind $(\mathcal{S}_2 - \ell) \leq 0$. Consequently, $\eta \notin \sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. The following requirement is proven $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) \leq 0$, assume $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) > 0$, then $\alpha(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$ and so $\beta(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$ thus $\eta \in \varphi(\mathcal{S}_1 \otimes \mathcal{S}_2)$. Let $\Lambda = \{(\hbar_i, \ell_i)_{i=1}^p \in \sigma(\mathcal{S}_1)\sigma(\mathcal{S}_2) : \hbar_i \ell_i = \eta\}$, where Λ is a finite set. And calculate $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta)$ we will use Theorem 3.5 in [10], whereas $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) = \sum_{j=n+1}^p \text{ind}(\mathcal{S}_1 - \hbar_j) \dim H_0(\mathcal{S}_2 - \ell_j) + \sum_{j=1}^n \text{ind}(\mathcal{S}_2 - \ell_j) \dim H_0(\mathcal{S}_1 - \hbar_j)$, since $\text{ind}(\mathcal{S}_1 - \hbar_i)$ and $\text{ind}(\mathcal{S}_2 - \ell_i)$ are non-positive, This is competitive. And so $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) \leq 0$ thus $\eta \notin \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

Rest to prove $\sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$. Let $\eta \notin \sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ then $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $p(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$ implies that $\eta \in \text{iso}\, \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$. For all factorization $\eta = \hbar\ell$ of η such that $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$ that is $\hbar \in \varphi_+(\mathcal{S}_1)$ and $\ell \in \phi_+(\mathcal{S}_2)$. As $\text{iso}\, \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \subset \text{iso}\, \sigma(\mathcal{S}_1)\, \text{iso}\, \sigma(\mathcal{S}_2)$, then $\mathcal{S}_1$ has $\mathcal{SVEP}$ at $\hbar$ and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at ℓ. Thus we have $p(\mathcal{S}_1 - \hbar) < \infty$ and $p(\mathcal{S}_2 - \ell) < \infty$, therefore $\hbar \notin \sigma_{ab}(\mathcal{S}_1)$ and $\ell \notin \sigma_{ab}(\mathcal{S}_2)$ and so $\eta \notin \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$.

We postulated $\eta \notin \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$, since $\eta \neq 0$for any factorization $\eta = \hbar\ell$ of η such that $\hbar \in \sigma(\mathcal{S}_1)$, $\ell \in \sigma(\mathcal{S}_2)$ and $\hbar \notin \sigma_{ab}(\mathcal{S}_1)$, $\ell \notin \sigma_{ab}(\mathcal{S}_2)$, then $\hbar \in \varphi_+(\mathcal{S}_1)$, $p(\mathcal{S}_1 - \hbar) < \infty$ and $\ell \in \varphi_+(\mathcal{S}_2)$, $p(\mathcal{S}_2 - \ell) < \infty$, implies that $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $\hbar \in \text{iso}\, \sigma(\mathcal{S}_1)$, $\ell \in \text{iso}\sigma\, (\mathcal{S}_2)$, that is $\eta \in \text{iso}\, \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$. It follows that $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $p(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$. Hence $\eta \notin \sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. So we get the result.

Definition 2.3 [3] A continuous linear operator $\mathcal{S} \in \mathcal{L}(\mathcal{A})$ is said to have property (ao) if $\sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \Pi_a(\mathcal{S})$.

Proposition 2.4 Let $\mathcal{S}$ be a continuous linear operators that the following are equivalent for $\mathcal{S}$

i- property (ao)holds for $\mathcal{S}$,

ii- $\sigma_{ab}(\mathcal{S}) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$.

Proof: For every operators $\mathcal{S}$, $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) \subseteq \sigma_{ab}(\mathcal{S})$. Let $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$, since property (ao) holds for $\mathcal{S}$ then $\eta \in \Pi_a(\mathcal{S})$. But by Theorem [3], property (Sab) holds for $\mathcal{S}$ then $\eta \in \Pi_a^0(\mathcal{S})$ while that $\Pi_a^0(\mathcal{S}) = \sigma_a(\mathcal{S}) \backslash \sigma_{ab}(\mathcal{S})$. Therefore $\sigma_{ab}(\mathcal{S}) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$.

Reciprocally, let $\eta \in \Pi_a(\mathcal{S})$, that is $\eta \in \sigma_a(\mathcal{S})$ and $\eta \notin \sigma_{LD}(\mathcal{S})$. But $\sigma_a(\mathcal{S}) \subseteq \sigma(\mathcal{S})$ and $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) \subseteq \sigma_{LD}(\mathcal{S})$, then we get $\eta \in \sigma(\mathcal{S})$ and $\eta \notin \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$. Thus $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$. Now, let $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$. Since $\sigma_{ab}(\mathcal{S}) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$ then property (az) holds for $\mathcal{S}$ and therefore $\eta \in \Pi_a^0(\mathcal{S})$. As $\Pi_a^0(\mathcal{S}) \subseteq \Pi(\mathcal{S})$, then $\eta \in \Pi_a(\mathcal{S})$. Consequently, property (ao) holds for $\mathcal{S}$.

The following Theorem proves that the above lemma validates for two directions if we add the condition $\mathcal{S}_1$ has property (ao) and $\mathcal{S}_2$ has property (ao).

Theorem 2.5 Suppose that $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$,and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, and both have property (ao), then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao) if and only if $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$.

Proof: We assume that $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao)then by above lemma we get the result.

Reciprocally, Since $\mathcal{S}_1$, $\mathcal{S}_2$ has property (ao) then $\sigma_{\mathrm{ab}}(\mathcal{S}_1) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)$, $\sigma_{\mathrm{ab}}(\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)$. According to the hypothesis

$$\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{\mathrm{ab}}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathrm{ab}}(\mathcal{S}_2)\sigma(\mathcal{S}_1) =$$

$\sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$, thus $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao).

Theorem 2.6 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ have property (ao). Then $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ if and only if $\mathcal{S}_1$ has $\mathcal{SVEP}$ at every points $\hbar \in \varphi_+(\mathcal{S}_1)$and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at every points $\ell \in \phi_+(\mathcal{S}_2)$ such that $0 \neq \eta = \hbar\ell \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

Proof: We assume that $\eta \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ then $\eta \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$, because $\mathcal{S}_1$ and $\mathcal{S}_2$ have property (ao). For every factorization $0 \neq \eta = \hbar\ell$ of η such that $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$, we have $\hbar \in \varphi_+(\mathcal{S}_1)$ and $\ell \in \varphi_+(\mathcal{S}_2)$ And consequently $p(\mathcal{S}_1 - \hbar) < \infty$ and $p(\mathcal{S}_2 - \ell) < \infty$. It leads to $\mathcal{S}_1$ has $\mathcal{SVEP}$ at $\hbar$ and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at ℓ.

Reciprocally, we must prove that $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$. Enough to prove that $\sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. Let $\eta \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ then $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $\mathrm{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2) \leq 0$. Hence for every factorization $0 \neq \eta = \hbar\ell$ of η where $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$, and $\hbar \in \varphi_+(\mathcal{S}_1)$, $\ell \in \varphi_+(\mathcal{S}_2)$. Since $\mathcal{S}_1$ has $\mathcal{SVEP}$ at $\hbar$ and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at ℓ then $p(\mathcal{S}_1 - \hbar) < \infty$ and $p(\mathcal{S}_2 - \ell) < \infty$. Therefore $\hbar \notin \sigma_{\mathrm{ab}}(\mathcal{S}_1)$ and $\ell \notin \sigma_{\mathrm{ab}}(\mathcal{S}_2)$. Thus $\eta \notin \sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

Theorem 2.7 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be continuous linear operator in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively. If $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao).

Proof: As $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$ then satisfy generalized a-Browder Theorem and consequently $\mathcal{S}_1$, $\mathcal{S}_2$ satisfy a-Browder Theorem. Then by Theorem 1 in [8], a-Browder Theorem holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$, Thus $\sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. It leads to property (ao)holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$.

Theorem 2.8 Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, If $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$ then $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ has property (ao).

Proof: As $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$ then we obtain by [1, corollary 3.73], $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ obey a-Browder Theorem. Consequently, $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ obey a-Browder Theorem. That is $\sigma_{\mathrm{ab}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*)$. Evidently, $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ obey property (ao).

Duggal in [5, 9]defined the polaroid operator as follows, if every isolated point of the spectrum of $\mathcal{S}$ is the pole of resolvent of $\mathcal{S}$, also η is pole of resolvent of $\mathcal{S}$ if and only if $0 < p(\mathcal{S} - \eta) = q(\mathcal{S} - \eta) < \infty$. Or equivalent, an operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$ is called polaroid if and only if there exists $\mathrm{d} = \mathrm{d}(\eta) \in \mathbb{N}$ such that $\mathrm{H}_0(\mathcal{S} - \eta) = \ker(\mathcal{S} - \eta)^{-1}$, for all $\eta \in \mathrm{iso}\sigma(\mathcal{S})$. Where $\mathrm{H}_0(\mathcal{S} - \eta)$ is a quasi-nilpotent part of $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$define as follows $\mathrm{H}_0(\mathcal{S} - \eta) = \{\mathrm{a} \in \mathrm{X}: \lim_{n\to\infty} \|(\mathcal{S} - \eta)^n \mathrm{a}\|^{\frac{1}{n}} = 0\}$.

Definition 2.9 [3] A continuous linear operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$is said to have property (SZ) if $\sigma(\mathcal{S}) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \mathrm{E}(\mathcal{S})$.

Theorem 2.10 Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$ are polaroid. If $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (SZ).

Proof: Let's start with the imposition $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$, then we have

$$\sigma_{\mathrm{W}}(\mathcal{S}_1) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1) = \sigma_{\mathrm{Bw}}(\mathcal{S}_1)$$

$$\sigma_{\mathrm{W}}(\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2) = \sigma_{\mathrm{Bw}}(\mathcal{S}_2),$$

also we have $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfies Browder's Theorem, thus

$$\sigma_{\mathrm{b}}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathrm{W}}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathrm{W}}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathrm{W}}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$$

$$= \sigma_{\mathrm{Bw}}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathrm{Bw}}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{\mathrm{Bw}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$$

$$= \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2).$$

As $\mathcal{S}_1$ and $\mathcal{S}_2$ are polaroid implies that $\mathcal{S}_1 \otimes \mathcal{S}_2$ is polaroid [6, Lemma 2], and consequently Weyl's Theorem holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$. From [7, Theorem 3.17], generalized Weyl Theorem holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$, thus $\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathrm{Bw}}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \mathrm{E}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. Plainly, $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (SZ).

Theorem 2.11 Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$ are polaroid. If $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$ then $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ has property (SZ).

Proof: We assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$, then we have from [1, corollary 2.5], [1, corollary 3.53], [2, Theorem 2.20]

$$\sigma_{\mathrm{W}}(\mathcal{S}_1{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^*) = \sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^*)$$

$$\sigma_{\mathrm{W}}(\mathcal{S}_2{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2{}^*) = \sigma_{\mathrm{Bw}}(\mathcal{S}_2{}^*),$$

also we have $\mathcal{S}_1{}^*$, $\mathcal{S}_2{}^*$ and $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ satisfies a-Browder's Theorem and therefore Browder's Theorem , thus

$$\sigma_{\mathrm{b}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*) = \sigma_{\mathrm{W}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*) = \sigma_{\mathrm{W}}(\mathcal{S}_1{}^*)\sigma(\mathcal{S}_2{}^*) \cup \sigma_{\mathrm{W}}(\mathcal{S}_2{}^*)\sigma(\mathcal{S}_1)$$

$$= \sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^*)\sigma(\mathcal{S}_2{}^*) \cup \sigma_{\mathrm{Bw}}(\mathcal{S}_2{}^*)\sigma(\mathcal{S}_1{}^*) = \sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*)$$

$$= \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^*)\sigma(\mathcal{S}_2{}^*) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2{}^*)\sigma(\mathcal{S}_1{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*).$$

As $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ are polaroid implies that $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ is polaroid [6, Lemma 2], and consequently Weyl's

Theorem holds for $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$. From [7, Theorem 3.17], generalized Weyl Theorem holds for $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$, thus $\sigma(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*)\backslash\sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*) = \sigma(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*)\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*) = \mathrm{E}(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*)$. Plainly, $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$ has property (SZ).

3. PERTURBATIONS

Assume $[\mathcal{Q},\mathcal{S}] = \mathcal{QS} - \mathcal{SQ}$ refer to the commutator of operators $\mathcal{Q},\mathcal{S} \in \mathrm{BL(X)}$. We assume that $\mathcal{Q}_1, \mathcal{Q}_2$ in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively, are a quasi-nilpotent operators $[\mathcal{Q}_1,\mathcal{S}_1] = [\mathcal{Q}_2,\mathcal{S}_2] = 0$ for some operators $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_2)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, hence $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2) = (\mathcal{S}_1\otimes\mathcal{S}_2) + \mathcal{Q}$, such that $\mathcal{Q} = \mathcal{S}_1\otimes\mathcal{Q}_1 + \mathcal{S}_2\otimes\mathcal{Q}_2 + \mathcal{Q}_1\otimes\mathcal{Q}_2 \in \mathrm{BL}(\mathrm{X}_1\otimes\mathrm{X}_2)$ is a quasi-nilpotent operator. Remember the definition of isoloid operator, $\mathcal{S} \in \mathrm{BL(X)}$, is isoloid if $\mathrm{iso}\,\sigma(\mathcal{S}) = \mathrm{E}(\mathcal{S})$.

Proposition 3.1 Suppose that $\mathcal{S} \in \mathrm{B(X)}$ be a polaroid operator then $\mathrm{E}(\mathcal{S}) = \Pi(\mathcal{S})$.

Proof: As always we have $\Pi(\mathcal{S}) \subseteq \mathrm{E}(\mathcal{S})$, for every operators $\mathcal{S}$. Now, let $\eta \in \mathrm{E}(\mathcal{S})$ that is $\eta \in \mathrm{iso}\,\sigma(\mathcal{S})$, since $\mathcal{S}$ is a polaroid then $\eta \in \Pi(\mathcal{S})$. Therefore $\mathrm{E}(\mathcal{S}) = \Pi(\mathcal{S})$.

Theorem 3.2 Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively, be a quasi-nilpotent operators $[\mathcal{Q}_1,\mathcal{S}_1] = [\mathcal{Q}_2,\mathcal{S}_2] = 0$ for some operators $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$. If $\mathcal{S}_1\otimes\mathcal{S}_2$ polaroid then property (ao) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$ implies $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)$ satisfies property (ao).

Proof: Observe that $\sigma(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, and that the perturbation of an operator by commuting quasi-nilpotent has $\mathcal{SVEP}$ if and only if the operator has $\mathcal{SVEP}$. If property (SZ) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$, hence

$\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2)$

$\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)\backslash\sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big) = \Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2)$,

we ought prove that $\Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2) = \Pi_{\mathrm{a}}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Let $\eta \in \Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2)$, it leads to $\eta \in \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$ and $\eta \in \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, also $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$. Clearly, if $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$ hence $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$ has $\mathcal{SVEP}$ at η and therefore $\Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2) = \Pi(\mathcal{S}_1\otimes\mathcal{S}_2)$,also we have $(\mathcal{S}_1{}^* + \mathcal{Q}_1{}^*)\otimes(\mathcal{S}_2{}^* + \mathcal{Q}_2{}^*)$ has $\mathcal{SVEP}$ at η, Implies that $\eta \in \mathrm{iso}\,\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Since $\mathcal{S}_1\otimes\mathcal{S}_2$ be a polaroid it leads to $\mathcal{S}_1\otimes\mathcal{S}_2$ an isoloid then $\eta \in \mathrm{E}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, consequently by above proposition we get $\eta \in \Pi\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Therefore, $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)$ satisfies property (ao).

Theorem 3.3 Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively, be a quasi-nilpotent operators $[\mathcal{Q}_1,\mathcal{S}_1] = [\mathcal{Q}_2,\mathcal{S}_2] = 0$ for some operators $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$. If $\mathcal{S}_1\otimes\mathcal{S}_2$ isoloid then property (SZ) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$ implies $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)$ satisfies property (SZ).

Proof: Observe that $\sigma(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, and that the perturbation of an operator by commuting quasi-nilpotent has $\mathcal{SVEP}$ if and only if the operator has $\mathcal{SVEP}$. If property (SZ) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$, hence

$\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2)$

$\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)\backslash\sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big) = \mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2)$,

rest we prove that $\mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2) = \mathrm{E}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Let $\eta \in \mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2)$, it leads to $\eta \in \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$ and $\eta \notin \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, also $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$. Clearly, if $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$ hence $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$ has $\mathcal{SVEP}$ at η and therefore $(\mathcal{S}_1{}^* + \mathcal{Q}_1{}^*)\otimes(\mathcal{S}_2{}^* + \mathcal{Q}_2{}^*)$ has $\mathcal{SVEP}$ at η. Implies that $\eta \in \mathrm{iso}\,\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Since $\mathcal{S}_1\otimes\mathcal{S}_2$ isoloid then $\eta \in \mathrm{E}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$.

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An ADMM for ARMA-State Space Model Estimation via Convex Optimization Using a Nuclear Norm Penalization Approach

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Abstract: Estimation of State-Space models together with joint model selection, is a difficult computational problem. Recent developments in convex penalization to least squares estimation problems provide an elegant solution to this problem that needs efficient optimization to be put to work in potentially large scale settings. In this paper, we study an Alternating Method of Multipliers for a penalized Subspace-type approach to State Space estimation with a nuclear norm penalty. Our model takes into account possible missing data. More-over, we show how creating artificial missing data at random provides a simple approach to hyper-parameter selection. Numerical experiments are proposed to illustrate the performance of our method.

Keywords: ARMA, Low Rank, Nuclear, Norm, Penalization.

1. Introduction

A real valued random discrete dynamical system $(x_t)_{t\in\mathbb{N}}$ admits a State Space representation if there exists a discrete time process $s_{t_{t\in\mathbb{N}}}$ such that

$$\begin{aligned} s_{t+1} &= As_t + Ke_t \\ x_t &= Bs_t + Ke_t \end{aligned}$$

Where $(e_t)_{t\in\mathbb{N}}$ is the noise, and $A\in\mathcal{R}^{p\times p}$, $B\in\mathcal{R}^{1\times p}$, $K\in\mathcal{R}^{p\times 1}$ are parameter matrices.

The Auto-regressive with moving average (ARMA) processes are sequences of the form $(x_t)_{t\in\mathbb{N}}$ that satisfy

$$x_t = \sum_{i=1}^{p} a_i x_{t-i} + \sum_{j=1}^{q} b_j e_{t-j} + e_t \qquad (1)$$

for all $t \geq \max\{p,q\}$, where $(e_t)_{t\in\mathbb{N}}$ is a sequence of independent identically distributed random variables. Time series model are relevant for a wide range of applications in economics, engineering, social science, epidemiology, ecology, signal processing,

It is well known that ARMA processes admit a State Space representation and vice versa [7, 4].

Time series analysis is concerned with two estimation problems.

The first is to select the orders p and q of the model.

The second is to estimate $a = (a_1, a_2, \dots, a_p)$ and $b = (b_1, b_2, \dots, b_q)$.

The model order selection problem is often performed using a penalized log-likelihood approach such as AIC, BIC,….

We refer the reader to the standard text of Shumway and Stoffer [7] for more details on this standard problems.

Turning to the estimation of a and b, it is well known that the log-likelihood is unfortunately not a concave function, and that multiple stationary points exist which can lead to severe bias when using local optimization routines for such as gradient or Newton-type methods for the joint estimation of a and b. In [7,3], an iterative procedure resembling the EM algorithm is proposed, which seems more appropriate for the ARMA model than standard optimization algorithms. However, no convergence grantee towards a global maximizer is provided. A recent advance in the field was the subspace method which turned out to be equivalent to minimizing a convex criterion for the estimation of a State Space model under stability conditions.

Since the recent successes of the LASSO in regression and its multiple generalizations [5], penalization has gained a lot of importance in computational statistics.

In particular, the nuclear norm has played an important role for many problems in engineering, machine learning and statistics such as matrix completion, ...

The goal of the present note is to study the nuclear norm penalization in the subspace method framework for convex minimization based ARMA estimation.

2. The Subspace Method

2.1 Prediction

The problem of predicting x_{t+j} for $j \geq 0$ based on the knowledge of $x_{t'}$, $t' < t$ and s_t can be solved easily following the approach by Bauer [2,8,1].

For given initial values x_0, e_0, the State Space representation gives

$$x_{t+h} = e_{t\_h} + \sum_{j=1}^{h} BA^{j-1}Ke_{t+h-j} + BA^h s_t$$

On the other hand, the State Space representation implies that

$$\begin{aligned} s_t &= As_{t-1} + Ke_{t-1} \\ &= As_{t-1} + K(x_{t-1} - Bs_{t-1}) \\ &= (A - KB)s_{t-1} + Kx_{t-1} \\ &= \cdots \end{aligned}$$

Thus, we obtain

$$s_t = (A - K\,B)^t\, s_0 + \sum_{j=0}^{t-1} (A - K\,B)^j\, K\, x_{t-1-j}$$

2.2 Prediction with Hankel matrices

We can rewrite the prediction problem in terms of some

Hankel matrices as explained in [6].
Define

$$\bar{A} = A - KB\ A_0 = [\bar{A}^t s_0, \bar{A}^{t+1} s_0, \dots, \bar{A}^{T-t+1} s_0]$$

$$K = [\bar{A}^{t-1}K, \dots, \bar{A}^2 K, K], \quad O = \begin{bmatrix} B \\ BA \\ \vdots \\ BA^{t-1} \end{bmatrix}$$

And

$$N = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ BK & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ BA^{t-2}K & BA^{t-3}K & \cdots & \cdots & BK & 1 \end{bmatrix}$$

Define also

$$X_{past} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{T-2t+1} \\ x_1 & x_2 & \cdots & x_{T-2t+2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{t-1} & x_t & \cdots & x_{T-t} \end{bmatrix},$$

$$X_{future} = \begin{bmatrix} x_t & x_{t+1} & \cdots & x_{T-t+1} \\ x_{t+1} & x_{t+2} & \cdots & x_{T-t+2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{2t-1} & x_{2t} & \cdots & x_T \end{bmatrix}$$

Both matrices are Hankel matrices. The first one represents the past values and and second one the future values.
Define also the noise matrix

$$E = \begin{bmatrix} e_t & e_{t+1} & \cdots & e_{T-t+1} \\ e_{t+1} & e_{e+2} & \cdots & e_{T-t+2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{2t-1} & e_{2t} & \cdots & e_T \end{bmatrix}$$

Now, as explained in [6], we have the following relationship

$$X_{future} = O\,K\,X_{past} + O\bar{A}_0 + NE \qquad (2)$$

3. The Estimation Problem

Using equation (2) , it is easy to build a least-squares estimator for the matrix L, [9].

In this section, we describe the nuclear norm-penalized estimator proposed in [6].

3.1 Estimating $O\,K$

The matrix $O\,K$ can be estimated using a least squares approach corresponding to solving

$$\min_{L \in R^{t \times t}} \frac{1}{2}\left\|X_{future} - X_{past}\right\|_F^2 \qquad (3)$$

This procedure will make sense if the term $O\bar{A}_0$ is small.
This can indeed be justified if t is large and if $\|\bar{A}\|$ is small.
Let us call $\hat{L}$ a solution of equation (3).

3.2 Nuclear Norm penalized least squares for low rank estimation

An interesting property of the matrix $O\,K$ is that its rank is the State's dimension p when A if full rank. Moreover, $O\,K$ has small rank compared to t when t is large compared to p.
Therefore, one is tempted to penalize the least squares problem in equation (3) with a low-rank promoting penalty. One option is to try to solve

$$\min_{L \in R^{t \times t}} \frac{1}{2}\left\|X_{future} - L\,X_{past}\right\|_F^2 + \lambda\,\mathrm{rank}(L) \qquad (4)$$

The main drawback of this approach is that the rank function is non continuous and non-convex function.
This renders the optimization problem intractable in practice. Fortunately, the rank function admits a well-known convex surrogate, which is the nuclear norm, i.e. the sum of the singular values, denoted by $\|.\|_*$.
Thus, a nice convex relaxation of (4) is given by

$$\min_{L \in R^{t \times t}} \frac{1}{2}\left\|X_{future} - LX_{past}\right\|_F^2 + \lambda\|L\|_* \qquad (5)$$

As is well known, the penalized least-squares problem (5) can be transformed into the following constrained problem

$$\min_{L \in R^{t \times t}} \|L\|_* \text{ subject to } \left\|X_{future} - LX_{past}\right\|_F \le \eta$$

for some appropriate choice of η.
The finite sample performance of this estimator was studied in [6].

3.3 The case of missing future data

The problem of handling missing data in the matrix X_{future} is easy to state. Let n_{obs} denote the number of observed entries in X_{future}.
Let $\Omega: R^{t \times T-2t+1} \to R^{n_{obs}}$ denote any operator of the user's choice which extracts the observed entries of X_{future} and stacks them into a real vector.
Then, based on the arguments of the previous section, a reasonable estimator can be proposed as the solution of

$$\min_{L \in R^{t \times t}} \frac{1}{2}\left\|\Omega(X_{future}) - \Omega(LX_{past})\right\|_F^2 + \lambda\|L\|_* \qquad (6)$$

for some appropriate choice of λ.

4. An ADMM for Computing $\hat{L}$

4.1 The standard case

Notice that equation (5) is equivalent to

$$\min \frac{1}{2}\left\|X_f - MX_p)\right\|_F^2 + \lambda\|L\|_*$$
$$\text{subject to } M = L$$

The Augmented Lagrange function is given by

$$L_p(M, L, U) = \frac{1}{2}\left\|X_f - MX_p\right\|_F^2 + \|L\|_* + \langle U, M - L\rangle + \frac{1}{2}\rho\|M - L\|_F^2$$

Minimize L_p for $M^{(l+1)}$ given $L^{(l)}$ and $U^{(l)}$, by finding the gradient of L_p with respect to M

$$\nabla_M L_p(M, L^{(l)}, U^{(l)}) = (X_f - MX_p)X_p^T + U^{(l)} + \rho(M - L^{(l)})$$

setting the gradient to 0 gives

$$\left(X_f - M^{(l+1)}X_p\right)X_p^T + U^{(l)} + \rho\left(M^{(l+1)} - L^{(l)}\right) = 0$$

Therefore,

$$X_f X_p^T - M^{(l+1)}X_p X_p^T + U^{(l)} + \rho M^{(l+1)} - \rho L^{(l)} = 0$$

$$X_f X_p^T + U^{(l)} - \rho L^{(l)} = M^{(l+1)}(X_f X_p^T - \rho I)$$

and thus

$$M^{(l+1)} = (X_f X_p^T + U^{(l)} - \rho L^{(l)})(X_p X_p^T - \rho I)^{-1}$$

Now, the next step is performed by computing the approximation of L by solving the following problem of minimization

$$\min_{L \in R^{t \times t}} \frac{1}{2}\rho\|L\|_F^2 - \rho\langle M, L\rangle - \langle U, L\rangle + \lambda\|L\|_*$$

$$= \min \frac{1}{2}\rho\|L\|_F^2 - \langle \rho M + U, L\rangle + \lambda\|L\|_*$$

$$= \min \frac{1}{2}\rho(\|L\|_F^2 - 2\langle M + \frac{1}{\rho}U, L\rangle) + \lambda\|L\|_*$$

$$= \min \frac{1}{2}\rho\left(\left\|L - (M + \frac{1}{\rho}U)\right\|_F^2\right) + \lambda\|L\|_*$$

Setting $Z = M + \frac{1}{\rho}U$, we obtain the optimization problem

$$= \min \frac{1}{2}\|L - Z\|_F^2 + \frac{1}{\rho}\lambda\|L\|_*$$

Thus, the solution is just defined by the thresholding operator as

$$L^{(l+1)} = \text{Thresh}(M^{(l+1)} + \frac{1}{\rho}U^{(l)}, \frac{\lambda}{\rho})$$

The last step consists in updating U, which is simply done by setting

$$U^{(l+1)} = U^{(l)} + \rho(M^{(l+1)} - L^{(l+1)}).$$

4.2 The case of missing data

Notice that equation (6) is equivalent to

$$\min \frac{1}{2}\left\|\Omega X_f - \Omega(MX_p)\right\|_2^2 + \lambda\|L\|_*$$

subject to M=L

The Augmented Lagrange function is given by

$$L_\rho(M, L, U) = \frac{1}{2}\left\|\Omega X_f - \Omega(MX_p)\right\|_2^2 + \lambda\|L\|_* + \langle U, M - L\rangle + \frac{1}{2}\rho\|M - L\|_F^2$$

Minimize L_ρ for $M^{(l+1)}$ given $L^{(l)}$ and $U^{(l)}$, by finding the gradient of L_ρ for M

$$\nabla_M L_\rho(M, L, U) = \Omega^*\left(\Omega(X_f) - \Omega(MX_p)\right)X_p^T + U + \rho(M - L)$$

setting the gradient to 0 gives

$$\Omega^*\left(\Omega(X_f) - \Omega(M^{(l+1)})X_p\right)X_p^T + U^{(l)} + \rho\left(M^{(l+1)} - L^{(L)}\right) = 0$$

Therefore, we obtain

$$\Omega^* \circ \Omega(X_f)X_p^T - \Omega^* \circ \Omega\left(M^{(l+1)}X_p\right)X_p^T + U^{(l)} + \rho M^{(l+1)} - \rho L^{(L)} = 0$$

Which gives

$$\Omega^* \circ \Omega(X_f)\,X_p^T + U^{(l)} - \rho L^{(L)} = \Omega^* \circ \Omega\left(M^{(l+1)}X_p\right)X_p^T + \rho M^{(l+1)}$$

This last equation may now be solved using the conjugate gradient method.

Now, the next step is performed exactly as in the previous case by computing the approximation of L by solving the following problem of minimization

$$\min_{L \in R^{txt}} \frac{1}{2}\rho\|L\|_F^2 - \rho\langle M^{(l+1)}, L\rangle - \langle U^{(l)}, L\rangle + \lambda\|L\|_*$$

whose solution is just defined by the thresholding operator as

$$L^{(L+1)} = \text{Thresh}(M^{(l+1)} + \frac{1}{\rho}U^{(l)}, \frac{\lambda}{\rho})$$

The last step consists in updating U, which is simply done by setting

$$U^{(l+1)} = U^{(l)} + \rho(M^{(l+1)} - L^{(l+1)})$$

5. Numerical Experiments

In this study we will perform some simulations with the model

$$x_t = 1.4x_{t-1} - .66x_{t-2} + .16x_{t-3} - .023x_{t-4} - .012x_{t-5} + e_t + 1.7e_{t-1} - 4e_{t-2} + 2.4e_{t-3} - .86e_{t-4}$$

with e_t, t=1,…,T independent zero mean Gaussian random variables with unit variance.

Figure.1 shows a realization of the signal considered in this section.

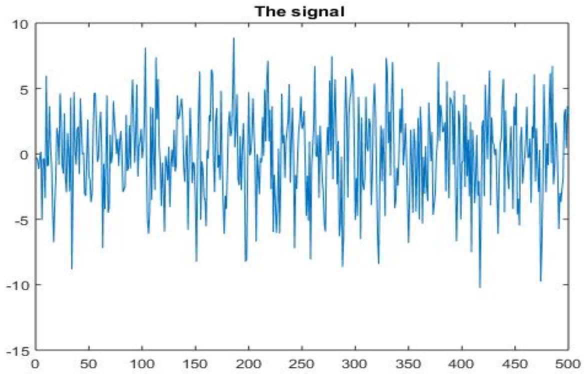

Figure 1. One realization of the signal

Figure.2 illustrates the convergence of the ADMM method.

In all experiments, the stopping criterion was when the relative error in the U variable went below 10^{-4}.

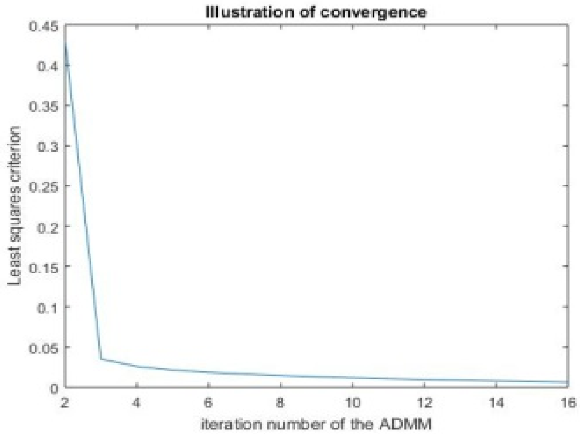

Figure 2. Decrease of the Least squares criterion as a function of the iteration number for 5 missing data and λ =20.

5.1 Choosing the relaxation parameter λ

A very simple way to choose the hyperparameter λ is to create artificially missing data in the set of future observations and tune the value of λ so as to minimize the sum of squares of the errors of the estimator on these observations. Figure.3 shows the error for different values of λ.

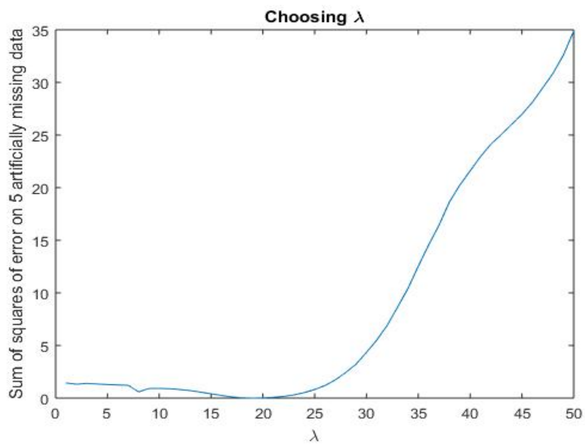

Figure 1. Error on the artificially missing data for selecting the best value for λ. Here, the best value is $\hat{\lambda} = 20$

Conclusion

The goal of the present paper was to present a nuclear norm penalised least-squares estimation procedure for ARMA model selection and estimation where the time series is corrupted by some noise and may have missing data. We proposed an ADMM type algorithm for this problem and studied the performances of the method on simulated data.

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Set-theoretical entropy of Alexandroff square homeomorphisms

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Abstract: In the following text for Alexandroff square $\mathbf{A}$,and unit square $\mathbf{O}$ (also equal to $[0,1]\times[0,1]$) equipped with lexicographic order topology if $X\in\{\mathbf{A},\mathbf{O}\}$ for homeomorphism $f:X\to X$ we have $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ moreover $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if f^4 is the identity map on X (where $\mathrm{ent}_{\mathrm{set}}(f)$ denotes set-theoretical entropy of f).

Keywords: Alexandroff square, lexicographic order, set-theoretical entropy**.**

1. Introduction

Several topologies have been introduced on unit square $[0,1]\times[0,1]$, like induced Euclidean topology, lexicographic order topology, Alexandroff square, etc.. In this text we consider $\mathbf{A}:=[0,1]\times[0,1]$ under topology generated by basis consisting of [3]:

- $\{t\}\times(U\setminus\{t\})$ where $t\in[0,1]$ and U is an open subset of $[0,1]$ (as a subset of real line $\mathbf{R}$),
- $([0,1]\setminus F)\times U$ where F is a finite subset of $[0,1]$ and U is an open subset of $[0,1]$ (as a subset of real line $\mathbf{R}$).
-

On the other hand several entropies have been introduced, e.g., topological entropy, algebraic entropy, adjoint entropy, set-theoretical entropy, etc.. Here we deal with set-theoretical entropy which has been introduced for the first time in [1]. For arbitrary set D, self-map $\lambda:D\to D$ and finite subset B of D the limit $h(B,\lambda):=\lim\limits_{n\to\infty}\frac{|B\cup\lambda(B)\cup\cdots\cup\lambda^{n-1}(B)|}{n}$ exists (where $|K|$ denotes the cardinality of finite set K). Define set-theoretical entropy of $\lambda:D\to D$ as $\sup\{h(F,\lambda):F$ is a finite subset of $D\}$ and denote it with $\mathrm{ent}_{\mathrm{set}}(\lambda)$.

In this text we compute all possible set-theoretical entropies of homeomorphism on Alexandroff square $\mathbf{A}$.

Remark 1.1. For $\lambda:D\to D$, $\mathrm{ent}_{\mathrm{set}}(\lambda)=\sup(\{n:$ there exist $x_1,\ldots,x_n\in D$ such that $\{\lambda^k(x_1)\}_{k\geq1},\ldots,\{\lambda^k(x_n)\}_{k\geq1}$ are n pairwise disjoint one-to-one sequences $\}\cup\{0\})$ [1]. Moreover for $t\geq1$ we have $\mathrm{ent}_{\mathrm{set}}(\lambda^t)=t\,\mathrm{ent}_{\mathrm{set}}(\lambda)$.

Convention 1.2. Using the same notations as in [2], by $<x,y>$ we mean ordered set $\{x,\{x,y\}\}$, and by (a,b) we mean open interval $\{z\in\mathbf{R}:a<z<b\}$, also in set $[0,1]\times[0,1]$, let $\Delta:=\{<t,t>:t\in[0,1]\}$ and:

$$\mathrm{P}_1:=<0,0>,\mathrm{P}_2:=<0,1>,\mathrm{P}_3:=<1,1>,\mathrm{P}_4:=<1,0>,$$
$$\mathrm{L}_1:=\{0\}\times(0,1),\mathrm{L}_2:=(0,1)\times\{1\},$$
$$\mathrm{L}_3:=\{1\}\times(0,1),\mathrm{L}_4:=(0,1)\times\{0\}.$$

2. Set-theoretical entropy of homeomorphisms of A

Lemma 2.1. For order preserving bijection $f:[0,1]\to[0,1]$ the following statements are equivalent:

- $\mathrm{ent}_{\mathrm{set}}(f)>0$,
- $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$,
- $f\neq\mathrm{id}_{[0,1]}$,

i.e., $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if $f=\mathrm{id}_{[0,1]}$.

Proof. Suppose $f\neq\mathrm{id}_{[0,1]}$, then there exists $t\in[0,1]$ with $f(t)\neq t$, without any loss of generality we may suppose $t<f(t)$ for $n\geq1$ choose $t=x_1<x_2<\cdots<x_n<f(t)$, then $t=x_1<x_2<\cdots<x_n<f(t)=f(x_1)<f(x_2)<\cdots<f(x_n)<f^2(x_1)<f^2(x_2)<\cdots<f^2(x_n)<\cdots$ and the sequences $\{f^k(x_1)\}_{k\geq1},\ldots,\{f^k(x_n)\}_{k\geq1}$ are pairwise disjoint and one-to-one, so by Remark 1.1 we have $\mathrm{ent}_{\mathrm{set}}(f)\geq n$. Hence $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$.

Remark 2.2. In Alexandroff square $\mathbf{A}$, for homeomorphism $f:\mathbf{A}\to\mathbf{A}$ we have $f(\Delta)=\Delta$ also for all $t\in[0,1]$ there exists $s\in[0,1]$ such that $f(\{t\}\times[0,1])=\{s\}\times[0,1]$ in addition $g:[0,1]\to[0,1]$ with $f<t,x>=<s,g(x)>$ is a homeomorphism. Moreover exactly one of the following conditions occurs [2]:

- $f(\mathrm{P}_i)=\mathrm{P}_i(i=1,2,3,4), f(\mathrm{L}_1)=\mathrm{L}_1, f(\mathrm{L}_3)=\mathrm{L}_3$,
- $f(\mathrm{P}_1)=\mathrm{P}_3,\quad f(\mathrm{P}_2)=\mathrm{P}_4,\quad f(\mathrm{P}_3)=\mathrm{P}_1,\quad f(\mathrm{P}_4)=\mathrm{P}_2$, $f(\mathrm{L}_1)=\mathrm{L}_3, f(\mathrm{L}_3)=\mathrm{L}_1$.

Theorem 2.3. In Alexandroff square $\mathbf{A}$, for homeomorphism $f:\mathbf{A}\to\mathbf{A}$ the following statements are equivalent:

- $\mathrm{ent}_{\mathrm{set}}(f)>0$,
- $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$,
- $f^4\neq \mathrm{id}_{\mathbf{A}}$,

i.e., $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if $f^4=\mathrm{id}_{\mathbf{A}}$.

Proof. Suppose $\mathrm{ent}_{\mathrm{set}}(f)>0$. By Remark 1.1, we have $\mathrm{ent}_{\mathrm{set}}(f^2)>0$. Moreover considering homeomorphism $f^2:\mathbf{A}\to\mathbf{A}$ by Remark 2.2 we have $f^2(\mathrm{P}_i)=\mathrm{P}_i$ $(i=1,2,3,4)$, also $f^2|_\Delta:\Delta\to\Delta$ is a homeomorphism. Note that Δ as a subspace of $\mathbf{A}$ has the same topology as a subspace of plane $\mathbf{R}^2$. Considering homeomorphism $h:[0,1]\to\Delta$ with $h(t)=<t,t>(t\in[0,1])$, we have homeomorphism $h^{-1}\circ f^2|_\Delta\circ h:[0,1]\to[0,1]$ with $(h^{-1}\circ f^2|_\Delta\circ h)(0)=(h^{-1}\circ f^2|_\Delta)(\mathrm{P}_1)=h^{-1}(\mathrm{P}_1)=0$ and $(h^{-1}\circ f^2|_\Delta\circ h)(1)=(h^{-1}\circ f^2|_\Delta)(\mathrm{P}_3)=h^{-1}(\mathrm{P}_3)=1$, so $h^{-1}\circ f^2|_\Delta\circ h:[0,1]\to[0,1]$ is an order preserving homeomorphism. Hence $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)\in\{0,+\infty\}$, by Lemma 2.1. We have the following cases:

- Case 1: $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)=+\infty$. By [1] we have $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)=\mathrm{ent}_{\mathrm{set}}(f^2|_\Delta)\leq\mathrm{ent}_{\mathrm{set}}(f^2)$, so in this case $\mathrm{ent}_{\mathrm{set}}(f^2)=+\infty$ which leads to $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$ by Remark 1.1.
- Case 2: $\mathrm{ent}_{\mathrm{set}}(h^{-1}\circ f^2|_\Delta\circ h)=0$. By Lemma 2.1, $h^{-1}\circ f^2|_\Delta\circ h=\mathrm{id}_{[0,1]}$ thus $f^2|_\Delta=\mathrm{id}_\Delta$. For all $t\in[0,1]$, by $f^2<t,t>=<t,t>$ and Remark 2.2 $g_t:[0,1]\to[0,1]$ with $f^2<t,x>=<t,g_t(x)>$ is a homeomorphism, hence $g_t^2:[0,1]\to[0,1]$ is an order preserving homeomorphism and $\mathrm{ent}_{\mathrm{set}}(g_t^2)\in\{0,+\infty\}$, using Lemma 2.1, we have the following sub-cases:
  - Sub-case 2-1: $\mathrm{ent}_{\mathrm{set}}(g_t^2)=0$ for all $t\in[0,1]$. By Lemma 2.1 for all $t\in[0,1]$ in this sub-case we have $g_t^2=\mathrm{id}_{[0,1]}$, thus for all $x\in[0,1]$ we have $f^4<t,x>=f^2<t,g_t(x)>=<t,g_t^2(x)>=<t,x>$, so in this sub-case $f^4=\mathrm{id}_{\mathbf{A}}$.
  - Sub-case 2-2: $\mathrm{ent}_{\mathrm{set}}(g_t^2)=+\infty$ for some $t\in[0,1]$. By Remark 1.1 for all $n\geq1$ there exist $x_1,\ldots,x_n\in[0,1]$ such that $\{g_t^{2k}(x_1)\}_{k\geq1},\ldots,\{g_t^{2k}(x_n)\}_{k\geq1}$ are n pairwise disjoint one-to-one sequences, however for all $k\geq1$ and $i\in\{1,\ldots,n\}$ we have $f^{2k}<t,x_i>=<t,g_t^{2k}(x_i)>$, thus
  $$\{f^{2k}<t,x_1>\}_{k\geq1},\ldots,\{f^{2k}<t,x_n>\}_{k\geq1}$$
  are n pairwise disjoint one-to-one sequences, so $\mathrm{ent}_{\mathrm{set}}(f^2)\geq n$ which leads to $\mathrm{ent}_{\mathrm{set}}(f^2)=+\infty$ and $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$ by Remark 1.1.

Using the above cases (and sub-cases) the proof is completed.

3. Set-theoretical entropy of homeomorphisms of lexicographic ordered unit square

Consider lexicographic order $\preceq$ on $[0,1]\times[0,1]$, such that for $<x,y>,<z,w>\in[0,1]\times[0,1]$, let $<x,y>\preceq<z,w>$ “$x<z$” or “$x=z$ and $y\leq w$”. Suppose $\mathbf{O}:=[0,1]\times[0,1]$ equipped with lexicographic order topology. In this section we compute set-theoretical entropies of homeomorphisms on $\mathbf{O}$.

Remark 3.1. In homeomorphism $f:\mathbf{O}\to\mathbf{O}$ for all $t\in[0,1]$ there exists $s\in[0,1]$ such that $f(\{t\}\times[0,1])=\{s\}\times[0,1]$ in addition $g:[0,1]\to[0,1]$ with $f<t,x>=<s,g(x)>$ is a homeomorphism. Moreover exactly one of the following conditions occurs [2]:

- $f(\mathrm{P}_i)=\mathrm{P}_i$, $f(\mathrm{L}_i)=\mathrm{L}_i(i=1,2,3,4)$, and $f:\mathbf{O}\to\mathbf{O}$ is order-preserving,
- $f(\mathrm{P}_1)=\mathrm{P}_3$, $f(\mathrm{P}_2)=\mathrm{P}_4$, $f(\mathrm{P}_3)=\mathrm{P}_1$, $f(\mathrm{P}_4)=\mathrm{P}_2$, $f(\mathrm{L}_1)=\mathrm{L}_3$, $f(\mathrm{L}_2)=\mathrm{L}_4$, $f(\mathrm{L}_3)=\mathrm{L}_1$, $f(\mathrm{L}_4)=\mathrm{L}_2$, and $f:\mathbf{O}\to\mathbf{O}$ is anti-order-preserving.

Theorem 3.2. For homeomorphism $f:\mathbf{O}\to\mathbf{O}$ the following statements are equivalent:

- $\mathrm{ent}_{\mathrm{set}}(f)>0$,
- $\mathrm{ent}_{\mathrm{set}}(f)=+\infty$,
- $f^2\neq\mathrm{id}_{\mathbf{O}}$,

i.e., $\mathrm{ent}_{\mathrm{set}}(f)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f)=0$ if and only if $f^2=\mathrm{id}_{\mathbf{O}}$.

Proof. Suppose $\mathrm{ent}_{\mathrm{set}}(f)>0$. By Remark 1.1, we have $\mathrm{ent}_{\mathrm{set}}(f^2)>0$. By Remark 3.1 for order-preserving homeomorphism $f^2:\mathbf{O}\to\mathbf{O}$ we have $f^2(\mathrm{P}_i)=\mathrm{P}_i$, $f^2(\mathrm{L}_i)=\mathrm{L}_i(i=1,2,3,4)$. Using similar method described in the proof of Lemma 2.1 we have: $\mathrm{ent}_{\mathrm{set}}(f^2)\in\{0,+\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(f^2)=0$ if and only if $f^2=\mathrm{id}_{\mathbf{O}}$. Use Remark 1.1 to complete the proof.

Example 3.3. Define $\varphi,\mu:[0,1]\to[0,1]$ with $\varphi(t):=1-t\,(t\in[0,1])$ and

$$\mu(t) := \begin{cases} 1-2t^2 & t \in [0, \frac{1}{2}], \\ \sqrt{\frac{1-t}{2}} & t \in [\frac{1}{2}, 1], \end{cases}$$

also consider $f, g : [0,1] \times [0,1] \to [0,1] \times [0,1]$ with $f < s,t >=< \varphi(s), \varphi(t) >$, $g < s,t >=< \mu(s), \mu(t) >$ (for $< s,t > \in [0,1] \times [0,1]$). Then:

- $f, g : \mathbf{A} \to \mathbf{A}$ and $f, g : \mathbf{O} \to \mathbf{O}$ are homeomorphisms,
- $f^2 = g^2 = \mathrm{id}_{[0,1] \times [0,1]}$ thus $\mathrm{ent}_{\mathrm{set}}(f) = \mathrm{ent}_{\mathrm{set}}(g) = 0$,
- $(g \circ f)^2(\frac{3}{4}) = \frac{31}{32}$ and $\mathrm{ent}_{\mathrm{set}}(g \circ f) = +\infty$ by Theorem 3.2.

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Generalized shift operators on ℓ^∞

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Abstract**:** In the following text we study the compactness of generalized shift operator on $\ell^\infty(\tau)$.

Keywords**:** Banach space, compact operator, generalized shift**.**

1. Introduction

One-sided shift $\underset{(x_1,x_2,\cdots)\mapsto(x_2,x_3,\cdots)}{\{1,\ldots,k\}^{\mathbf{N}}\to\{1,\ldots,k\}^{\mathbf{N}}}$ and two-sided shift $\underset{(x_n)_{n\in\mathbf{Z}}\mapsto(x_{n+1})_{n\in\mathbf{Z}}}{\{1,\ldots,k\}^{\mathbf{Z}}\to\{1,\ldots,k\}^{\mathbf{Z}}}$ are amongst most studied maps [6]. Consider arbitrary sets A,Γ with at least two elements and $\varphi:\Gamma\to\Gamma$, we call $\sigma_\varphi:A^\Gamma\to A^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma})=(x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ $((x_\alpha)_{\alpha\in\Gamma}\in A^\Gamma)$ a generalized shift (as a generalization of one-sided and two-sided shifts) which has been introduced for the first time in [2]. Dynamical and non-dynamical properties of generalized shifts have been studied in several texts like [3, 5].

It is well-known that for each (complex) Hilbert space H there exists a unique cardinal number τ such that H and $\ell^2(\tau)=\{(x_\alpha)_{\alpha<\tau}\in\mathbf{C}^\tau:\sum\limits_{\alpha<\tau}|x_\alpha|^2<+\infty\}$ (equipped with inner product $<(x_\alpha)_{\alpha<\tau},(y_\alpha)_{\alpha<\tau}>=\sum\limits_{\alpha<\tau}x_\alpha\overline{y_\alpha}$ and norm $\|(x_\alpha)_{\alpha<\tau}\|=\sqrt{\sum\limits_{\alpha<\tau}|x_\alpha|^2}$), where $\mathbf{C}$ denotes the field of complex numbers. So for $\varphi:\tau\to\tau$ one may consider $\sigma_\varphi|_{\ell^2(\tau)}:\ell^2(\tau)\to\mathbf{C}^\tau$. As it has mentioned in [1], the following statements are equivalent (note that $\sigma_\varphi:\mathbf{C}^\tau\to\mathbf{C}^\tau$ is a linear map):

- $\sigma_\varphi|_{\ell^2(\tau)}(\ell^2(\tau))\subseteq\ell^2(\tau)$,
- $\sigma_\varphi|_{\ell^2(\tau)}(\ell^2(\tau))\subseteq\ell^2(\tau)$ and $\sigma_\varphi|_{\ell^2(\tau)}:\ell^2(\tau)\to\ell^2(\tau)$ is continuous,
- $\varphi:\tau\to\tau$ is bounded, i.e., there exists $K\in\mathbf{N}$ such that for all $\alpha\in\tau$ the set $\varphi^{-1}(\alpha)$ has at most K elements.

In the following text we consider the following Banach space (equipped with norm $\|(x_\alpha)_{\alpha<\tau}\|_\infty=\sup\limits_{\alpha<\tau}|x_\alpha|$):

$$\ell^\infty(\tau)=\{(x_\alpha)_{\alpha<\tau}\in\mathbf{C}^\tau:\sup_{\alpha<\tau}|x_\alpha|<+\infty\}$$

we study $\sigma_\varphi|_{\ell^\infty(\tau)}$.

2. Results on $\sigma_\varphi|_{\ell^\infty(\tau)}$

In this section suppose $\tau\geq 2$ is a cardinal number and $\varphi:\tau\to\tau$ is arbitrary, as our first steps we prove the following theorem.

Theorem 1. We have the following statements:

a. $\sigma_\varphi(\ell^\infty(\tau))\subseteq\ell^\infty(\tau)$,

b. $\sigma_\varphi|_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is continuous and (note that $\|\sigma_\varphi|_{\ell^\infty(\tau)}\|=\sup\{\|\sigma_\varphi(z)\|_\infty:z\in\ell^\infty(\tau),\|z\|_\infty\leq 1\}$):

$$\|\sigma_\varphi|_{\ell^\infty(\tau)}\|=1,$$

c. the following statements are equivalent:

1. $\sigma_\varphi(\ell^\infty(\tau))=\ell^\infty(\tau)$,
2. $\sigma_\varphi(\ell^\infty(\tau))$ is dense in $\ell^\infty(\tau)$,
3. $\varphi:\tau\to\tau$ is one-to-one.

Proof. a, b) Consider $x=(x_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$, then

$$\|\sigma_\varphi(x)\|_\infty=\|\sigma_\varphi((x_\alpha)_{\alpha<\tau})\|_\infty=\|(x_{\varphi(\alpha)})_{\alpha<\tau}\|_\infty$$
$$=\sup_{\alpha<\tau}|x_{\varphi(\alpha)}|\leq\sup_{\alpha<\tau}|x_\alpha|=\|(x_\alpha)_{\alpha<\tau}\|_\infty=\|x\|_\infty$$

and $\|\sigma_\varphi(x)\|_\infty\leq\|x\|_\infty$, hence $\sigma_\varphi(x)\in\ell^\infty(\tau)$, also $\sigma_\varphi|_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is continuous and $\|\sigma_\varphi|_{\ell^\infty(\tau)}\|\leq 1$, on the other hand $(1)_{\alpha<\tau}\in\ell^\infty(\tau)$ and $\|\sigma_\varphi((1)_{\alpha<\tau})\|_\infty=\|(1)_{\alpha<\tau}\|_\infty=1$ which completes the proof of $\|\sigma_\varphi|_{\ell^\infty(\tau)}\|=1$.

c) We complete the proof by showing "$(2)\Rightarrow(3)$" and "$(3)\Rightarrow(1)$".

$(2)\Rightarrow(3)$: Suppose $\varphi:\tau\to\tau$ is not one-t-one, choose $\beta<\theta<\tau$ with $\varphi(\beta)=\varphi(\theta)$. Let $q_\beta=1$ and $q_\alpha=0$ for $\alpha\neq\beta$. Then $U:=\{x\in\ell^\infty(\tau):\|x-(q_\alpha)_{\alpha<\tau}\|_\infty<\frac{1}{2}\}$ is an open neighborhood of $(q_\alpha)_{\alpha<\tau}(\in\ell^\infty(\tau))$, moreover for all $(x_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$ we have

$$\|\sigma_\varphi(x)-(q_\alpha)_{\alpha<\tau}\|_\infty=\|(x_{\varphi(\alpha)})_{\alpha<\tau}-(q_\alpha)_{\alpha<\tau}\|_\infty$$
$$=\sup_{\alpha<\tau}|x_{\varphi(\alpha)}-q_\alpha|\geq\max(|x_{\varphi(\beta)}-q_\beta|,|x_{\varphi(\theta)}-q_\theta|)$$
$$=\max(|x_{\varphi(\beta)}-1|,|x_{\varphi(\theta)}|)\geq\tfrac{1}{2}(|x_{\varphi(\beta)}-1|+|x_{\varphi(\theta)}|)$$
$$\overset{\varphi(\beta)=\varphi(\theta)}{=}\tfrac{1}{2}(|x_{\varphi(\beta)}-1|+|x_{\varphi(\beta)}|)\geq\tfrac{1}{2}|x_{\varphi(\beta)}-1-x_{\varphi(\beta)}|=\tfrac{1}{2}$$

thus $\sigma_\varphi(\ell^\infty(\tau))\cap U$ is empty and $\sigma_\varphi(\ell^\infty(\tau))$ is not dense in $\ell^\infty(\tau)$.

(3)$\Rightarrow$(1): Suppose $\varphi:\tau\to\tau$ is one-to-one and choose $x=(x_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$ define $y=(y_\alpha)_{\alpha<\tau}$ with:

$$y_\alpha:=\begin{cases}x_\beta & \beta<\tau,\alpha=\varphi(\beta),\\ 0 & \text{otherwise}.\end{cases}$$

Then

$$\|y\|_\infty=\sup_{\alpha<\tau}|y_\alpha|=\sup_{\substack{\alpha=\varphi(\beta),\\ \beta<\tau}}|x_\beta|\le\sup_{\alpha<\tau}|x_\alpha|=\|x\|_\infty<+\infty$$

and $y\in\ell^\infty(\tau)$. Moreover $\sigma_\varphi(y)=(y_{\varphi(\alpha)})_{\alpha<\tau}=(x_\alpha)_{\alpha<\tau}$ which completes the proof.

Let's recall that in Banach spaces X,Y we say linear continuous map $T:X\to Y$ is a compact operator if $\overline{\{T(x):\|x\|<1\}}$ is a compact subset of Y [4].

Theorem 2. $\sigma_\varphi|_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is a compact operator if and only if $\varphi(\tau)$ is finite.

Proof. First suppose $\varphi(\tau)$ is infinite. Choose one-to-one sequence $\{\alpha_i\}_{i\ge1}$ in τ such that $\{\varphi(\alpha_i)\}_{i\ge1}$ is a one-to-one sequence too. For each $i\ge1$ let $x_i=(x^i_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$ with $x^i_{\alpha_i}=\frac12$ and $x^i_\alpha=0$ for $\alpha\ne\alpha_i$. Then for $i\ne j$ we have $\|\sigma_\varphi(x_i)-\sigma_\varphi(x_j)\|_\infty=\frac12$ and $\{\sigma_\varphi(x_i)\}_{i\ge1}$ does not have any convergent subsequence however $\{x_i\}_{i\ge1}$ is a sequence in $\{x\in\ell^\infty(\tau):\|x\|_\infty<1\}$, so $\sigma_\varphi|_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is not compact.

Now suppose $\varphi(\tau)$ is finite, in this case $\sigma_\varphi(\ell^\infty(\tau))$ is a finite dimensional subset of $\sigma_\varphi(\ell^\infty(\tau))$, thus its closed bounded subsets are compact, using Theorem 1, $\sigma_\varphi\{x\in\ell^\infty(\tau):\|x\|_\infty<1\}(\subseteq\{x\in\ell^\infty(\tau):\|x\|_\infty<1\})$ is a bounded subset of $\sigma_\varphi(\ell^\infty(\tau))$, which leads to the desired result.

3. Generalized shifts on subspaces of ℓ^∞

As it is common in the literature, for the least infinite cardinal number $\omega=\{0,1,2,\dots\}$ we denote $\ell^\infty(\omega)$ by ℓ^∞. Consider the following subspaces of ℓ^∞:

- $\ell^\infty_{00}:=\{(x_n)_{n<\omega}\in\ell^\infty:\exists N\ \forall n\ge N\ x_n=0\}$
- $\ell^\infty_{0c}:=\{(x_n)_{n<\omega}\in\ell^\infty:\exists z\ \exists N\ \forall n\ge N\ x_n=z\}$
- $\ell^\infty_{0}:=\{(x_n)_{n<\omega}\in\ell^\infty:\lim_{n\to+\infty}x_n=0\}$
- $\ell^\infty_{c}:=\{(x_n)_{n<\omega}\in\ell^\infty:\exists z\lim_{n\to+\infty}x_n=0\}$

thus $\ell^\infty_{00}\subseteq\ell^\infty_0\subseteq\ell^\infty_c\subseteq\ell^\infty$ and $\ell^\infty_{00}\subseteq\ell^\infty_{0c}\subseteq\ell^\infty_c\subseteq\ell^\infty$. In this section consider $\varphi:\omega\to\omega$.

Theorem 3. The following statements are equivalent:

1. $\sigma_\varphi(\ell^\infty_{00})\subseteq\ell^\infty_{00}$,
2. $\sigma_\varphi(\ell^\infty_{0})\subseteq\ell^\infty_{0}$,
3. for all $n\in\omega$ the set $\varphi^{-1}(n)$ is finite (i.e., φ is finite fiber).

Proof. "(2)$\Rightarrow$(3)" and "(1)$\Rightarrow$(3)": Suppose there exists $p\in\omega$ such that $\varphi^{-1}(p)$ is infinite. Consider $u=(u_n)_{n<\omega}$ with $u_p=1$ and $u_n=0$ for $n\ne p$. Then we have $u\in\ell^\infty_{00}(=\ell^\infty_0\cap\ell^\infty_{00})$ and $\sigma_\varphi(u)\notin\ell^\infty_0(=\ell^\infty_0\cup\ell^\infty_{00})$, thus not only $\sigma_\varphi(\ell^\infty_0)\not\subset\ell^\infty_0$, but also $\sigma_\varphi(\ell^\infty_{00})\not\subset\ell^\infty_{00}$.

(3)$\Rightarrow$(1): Suppose (3) is valid and $(x_n)_{n<\omega}\in\ell^\infty_{00}$, then there exists $N\in\omega$ such that for all $n\ge N$ we have $x_n=0$. Since φ is finite fiber, $\varphi^{-1}(\{0,\dots,N\})$ is finite and $m=\max(\varphi^{-1}(\{0,\dots,N\})\cup\{0\})\in\omega$. So $x_{\varphi(n)}=0$ for all $n\ge m+1$. Hence $\sigma_\varphi((x_n)_{n<\omega})=(x_{\varphi(n)})_{n<\omega}\in\ell^\infty_{00}$.

(3)$\Rightarrow$(2): Suppose (3) is valid and $(x_n)_{n<\omega}\in\ell^\infty_0$, then $\lim_{n\to+\infty}x_n=0$ and for every $\varepsilon>0$ there exists $N\in\omega$ such that for all $n\ge N$ we have $|x_n|<\varepsilon$. Since φ is finite fiber, $m=\max(\varphi^{-1}(\{0,\dots,N\})\cup\{0\})\in\omega$. So for all $n\ge m+1$ we have $|x_{\varphi(n)}|<\varepsilon$. Thus $\lim_{n\to+\infty}x_{\varphi(n)}=0$ and $\sigma_\varphi((x_n)_{n<\omega})=(x_{\varphi(n)})_{n<\omega}\in\ell^\infty_0$.

Theorem 4. The following statements are equivalent:

1. $\sigma_\varphi(\ell^\infty_{0c})\subseteq\ell^\infty_{0c}$,
2. $\sigma_\varphi(\ell^\infty_{c})\subseteq\ell^\infty_{c}$,
3. for all $n\in\omega$ "$\varphi^{-1}(n)$ is finite" or "$\omega\setminus\varphi^{-1}(n)$ is finite".

Proof. First suppose there exists $p\in\omega$ such that both sets $\varphi^{-1}(p)$ and $\omega\setminus\varphi^{-1}(p)$ are infinite. Consider $u=(u_n)_{n<\omega}$ with $u_p=1$ and $u_n=0$ for $n\ne p$. Then we have $u\in\ell^\infty_{0c}(=\ell^\infty_c\cap\ell^\infty_{0c})$, let $(v_n)_{n<\omega}=(u_{\varphi(n)})_{n<\omega}=\sigma_\varphi(u)$. Using infiniteness of $\varphi^{-1}(p)$ and $\omega\setminus\varphi^{-1}(p)$ there exist $m_1<m_2<\cdots$ in $\varphi^{-1}(p)$ and there exist $k_1<k_2<\cdots$ in $\omega\setminus\varphi^{-1}(p)$ thus $\lim_{n\to\infty}v_{m_n}=1$ and $\lim_{n\to\infty}v_{k_n}=0$. Hence $\lim_{n\to\infty}v_n$ does not exist and $\sigma_\varphi(u)=(v_n)_{n<\omega}\notin\ell^\infty_c$. So not only $\sigma_\varphi(\ell^\infty_0)\not\subset\ell^\infty_0$, but also $\sigma_\varphi(\ell^\infty_{00})\not\subset\ell^\infty_{00}$. Thus "(2)$\Rightarrow$(3)" and "(1)$\Rightarrow$(3)".

(3) $\Rightarrow$ (1): Suppose (3) is valid and $(x_n)_{n<\omega} \in \ell^{\infty}_{0c}$, then there exists $N \in \omega$ such that for all $n \ge N$ we have $x_n = x_N =; z$. We have the following cases:

Case 1: φ is finite fiber. In this case $\varphi^{-1}(\{0,\dots,N\})$ is finite and $m = \max(\varphi^{-1}(\{0,\dots,N\}) \cup \{0\}) \in \omega$. So $x_{\varphi(n)} = z$ for all $n \ge m+1$. Hence $\sigma_{\varphi}((x_n)_{n<\omega}) = (x_{\varphi(n)})_{n<\omega} \in \ell^{\infty}_{0c}$.

Case 2: there exists $p \in \omega$ such that $\varphi^{-1}(p)$ is infinite. So in this case $\omega \setminus \varphi^{-1}(p)$ is finite and there exists $M \in \omega$ with $\omega \setminus \varphi^{-1}(p) \subseteq \{0,\dots,M\}$. For all $n \ge M+1$ we have $n \in \varphi^{-1}(p)$ and $\varphi(n) = p$, hence $x_{\varphi(n)} = x_p$ which shows $\sigma_{\varphi}((x_n)_{n<\omega}) = (x_{\varphi(n)})_{n<\omega} \in \ell^{\infty}_{0c}$.

(3) $\Rightarrow$ (2): Suppose (3) is valid and $(x_n)_{n<\omega} \in \ell^{\infty}_{c}$, then $\{x_n\}_{n<\omega}$ is a convergent and hence Cauchy so for every $\varepsilon > 0$ there exists $N \in \omega$ such that for all $n, m \ge N$ we have $|x_n - x_m| < \varepsilon$. We have the following cases:

Case 1: φ is finite fiber. In this case $\varphi^{-1}(\{0,\dots,N\})$ is finite and $M = \max(\varphi^{-1}(\{0,\dots,N\}) \cup \{0\}) \in \omega$. So for all $n, m \ge M+1$ we have $\varphi(n), \varphi(m) > N$ therefore $|x_{\varphi(n)} - x_{\varphi(m)}| < \varepsilon$.

Case 2: there exists $p \in \omega$ such that $\varphi^{-1}(p)$ is infinite. So in this case $\omega \setminus \varphi^{-1}(p)$ is finite and there exists $M \in \omega$ with $\omega \setminus \varphi^{-1}(p) \subseteq \{0,\dots,M\}$. For all $n, m \ge M+1$ we have $x_{\varphi(n)} = x_p = x_{\varphi(m)}$ which shows $|x_{\varphi(n)} - x_{\varphi(m)}| = 0 < \varepsilon$.

Using the above cases, there exists $M \in \omega$ with $|x_{\varphi(n)} - x_{\varphi(m)}| < \varepsilon$ for all $n, m \ge M+1$. Therefore $\{x_{\varphi(n)}\}_{n<\omega}$ is a Cauchy hence convergent sequence in $\mathbf{C}$. Therefore $\sigma_{\varphi}((x_n)_{n<\omega}) = (x_{\varphi(n)})_{n<\omega} \in \ell^{\infty}_{c}$.

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Study of a forwarding chain with respect to operators in the Self-maps sub-category

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Abstract: In the following chain we study some backwarding, forwarding and stationary chains in the category Set with respect to some well-known operators like composition, finite product and disjoint union.
***Keywords*:** backwarding chain, forwarding chain, stationary chain.

1. Introduction

Our main aim in this text is to study the concept of forwarding (backwarding, stationary) chain in sub-categories of Self-maps in category Set.

In the category C suppose M is a nonempty chain of sub-categories of C (under the inclusion relation, so elements of M are sub-categories of C and for each $\alpha,\beta\in\mathrm{M}$ we have $\alpha\subseteq\beta$ or $\beta\subseteq\alpha$ (since M is a chain)). Also suppose $h:\bigcup\mathrm{M}\to\bigcup\mathrm{M}$ is a map. We say M is [1]:

- a forwarding chain with respect to h if for all $\kappa\in\mathrm{M}$ we have $h(\bigcup\mathrm{M}\setminus\kappa)\subseteq\bigcup\mathrm{M}\setminus\kappa$ (i.e., $h(\bigcup\mathrm{M}\setminus\kappa)\cap\kappa$ is empty),
- a full-forwarding chain with respect to h if it is forwarding and for all distinct $\kappa,\lambda,\mu\in\mathrm{M}$ with $\kappa\subseteq\lambda\subseteq\mu$ there exists $X\in\lambda\setminus\kappa$ with $h(X)\in\mu\setminus\lambda$,
- a backwarding chain with respect to h if for all $\kappa\in\mathrm{M}$ we have $h(\kappa)\subseteq\kappa$,
- a full-backwarding chain with respect to h if it is backwarding and for all distinct $\kappa,\lambda,\mu\in\mathrm{M}$ with $\kappa\subseteq\lambda\subseteq\mu$ there exists $X\in\mu\setminus\lambda$ with $h(X)\in\lambda\setminus\kappa$,
- a stationary chain with respect to h if it is both forwarding and backwarding chain with respect to h.

Let's recall that for equivalence relation E on X and $x\in X$ we have $\frac{x}{E}:=\{y\in X:(x,y)\in E\}$ and quotient space $\frac{X}{E}:=\left\{\frac{z}{E}:z\in X\right\}$. Also $\aleph_0$ denotes the least infinite cardinal number, i.e., $\mathrm{card}(\mathbf{N})=\aleph_0$ (where $\mathbf{N}$ is the collection of all natural numbers).

For self-map $f:X\to X$ consider two equivalence relations $\Im_f$ and $\Re_f$ on X with (see e.g. [2]):

$$(x,y)\in\Im_f\Leftrightarrow f(x)=f(y),$$

$$(x,y)\in\Re_f\Leftrightarrow(\exists n,m\geq 1\;\; f^n(x)=f^m(y)).$$

In this text for cardinal number $\tau>1$ suppose:

- $D_\tau:=\{X\overset{f}{\to}X:$ cardinality of the quotient space $\frac{X}{\Im_f}$ is less than $\tau\}$,
- $E_\tau:=\{X\overset{f}{\to}X:$ cardinality of the quotient space $\frac{X}{\Re_f}$ is less than $\tau\}$.

We denote the sub-category of Set consisting of self-maps by SSet and will denote self-map $f:X\to X$ by (X,f).

2. First operator: *k* times self-composition

In this section consider $k\geq 2$ and $h_1:\mathrm{SSet}\to\mathrm{SSet}$ with $h_1(X,f)=(X,f^k)$ (where $f^k=f\circ\cdots\circ f$ (k times)).

Lemma 1. For $(X,f)\in\mathrm{SSet}$ we have $\Im_f\subseteq\Im_{f^k}$ and $\Re_f=\Re_{f^k}$, thus $\mathrm{card}(\frac{X}{\Im_{f^k}})\leq\mathrm{card}(\frac{X}{\Im_f})$ and $\mathrm{card}(\frac{X}{\Re_{f^k}})=\mathrm{card}(\frac{X}{\Re_f})$.

Proof. For each $(X,f)\in\mathrm{SSet}$ and $(x,y)\in\Im_f$ we have $f(x)=f(y)$ thus $f^k(x)=f^k(y)$ and $(x,y)\in\Im_{f^k}$, therefore $\Im_f\subseteq\Im_{f^k}$ and

$$\begin{array}{c}\frac{X}{\Im_f}\to\frac{X}{\Im_{f^k}}\\ \frac{z}{\Im_f}\mapsto\frac{z}{\Im_{f^k}}\end{array}$$

is onto, hence $\mathrm{card}(\frac{X}{\Im_{f^k}})\leq\mathrm{card}(\frac{X}{\Im_f})$. Moreover, $x,y\in X$ we have:

$(x,y)\in\Re_f\Leftrightarrow\exists n,m\geq 1(f^m(x)=f^n(y))$

$$\Leftrightarrow \exists n,m \geq 1\,(f^{mk}(x) = f^{nk}(y))$$
$$\Leftrightarrow \exists n,m \geq 1\,((f^{k})^{m}(x) = (f^{k})^{n}(y))$$
$$\Leftrightarrow (x,y) \in \Re_{f^k}.$$

Which leads to $\Re_f = \Re_{f^k}$ and completes the proof.

Theorem 2. Consider nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{D_\tau : \tau > 1\}$:

a. M is backwarding with respect to h_1.

b. M is forwarding (resp. stationary) with respect to h_1 iff M is singleton,

Proof. (a) By Lemma 1, $h_1(D_\tau) \subseteq D_\tau$ for each $\tau > 1$, thus $h_1(\bigcup\mathrm{M}) \subseteq \bigcup\mathrm{M}$ and M is backwarding with respect to h_1.

(b) Now suppose M has at least two elements and consider distinct elements $H, K \in \mathrm{M}$ with $H \subset K$. There exists $\tau > 1$ with $H = D_\tau$. Choose cardinal number $\theta > 0$ with $\tau = \theta + 1$. Consider arbitrary set A with $\mathrm{card}(A) = \theta$ and $b \notin A \times \{0,1\}$ (e.g., $b = (0,-1)$). Let $X = (A \times \{0,1\}) \cup \{b\}$ and define $f : X \to X$ with $f(a,0) = (a,1)$, $f(a,1) = b$ and $f(b) = b$. Then

$$\frac{X}{\Im_f} = \{\{(a,0)\} : a \in A\} \cup \{(A \times \{1\}) \cup \{b\}\}$$

and $\mathrm{card}(\frac{X}{\Im_f}) = \theta + 1 = \tau$. Thus $(X, f) \notin D_\tau = H$ and for each $\psi > \tau$ we have $(X, f) \in D_\psi \subset \mathrm{SSet}$, in particular $(X, f) \in K \setminus C \subseteq \bigcup\mathrm{M} \setminus C$. On the other hand $\frac{X}{\Im_{f^k}} = \{X\}$, hence $h_1(X, f) = (X, f^k) \in D_2 \subseteq D_\tau = C$.

Therefore, M is not forwarding (resp. stationary) with respect to h_1.

Corollary 3. Each nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau > 1\}$, is stationary (resp. forwarding, backwarding) with respect to h_1.

Proof. Use Lemma 1.

3. Second operator: finite *k* times self-product

For $k \geq 2$ consider $h_2 : \mathrm{SSet} \to \mathrm{SSet}$ with $h_2(X, f) = (X^k, f_k)$, $f_k(y_1,\cdots,y_k) = (f(y_1),\cdots, f(y_k))$.

Lemma 4. Consider $(X, f) \in \mathrm{SSet}$:

1. we have:

$$\mathrm{card}(\frac{X^k}{\Im_{f_k}}) = \left(\mathrm{card}(\frac{X}{\Im_f})\right)^k \begin{cases} < \aleph_0 & \frac{X}{\Im_f} \text{ is finite}, \\ = \mathrm{card}(\frac{X}{\Im_f}) & \text{otherwise}. \end{cases}$$

In particular for $\tau \in \{\theta : \theta = 2 \vee \theta \geq \aleph_0\}$, $(X, f) \in D_\tau$ iff $h_2(X, f) \in D_\tau$.

2. we have:

$$\mathrm{card}(\frac{X}{\Re_f}) \leq \mathrm{card}(\frac{X^k}{\Re_{f_k}}) \leq \left(\mathrm{card}(\frac{X}{\Re_f})\right)^k.$$

In particular for $\tau \in \{\theta : \theta = 2 \vee \theta \geq \aleph_0\}$, $(X, f) \in E_\tau$ iff $h_2(X, f) \in E_\tau$.

Proof. (1) For $(x_1,\cdots,x_k),(y_1,\cdots,y_k) \in X^k$ we have:

$((x_1,\cdots,x_k),(y_1,\cdots,y_k)) \in \Im_{f_k}$

$$\Leftrightarrow f_k(x_1,\cdots,x_k) = f_k(y_1,\cdots,y_k)$$
$$\Leftrightarrow (f(x_1),\cdots, f(x_k)) = (f(y_1),\cdots, f(y_k))$$
$$\Leftrightarrow (x_1,y_1),\ldots,(x_k,y_k) \in \Im_f$$

so

$$\underset{\frac{(z_1,\cdots,z_k)}{\Im_{f_k}} \mapsto (\frac{z_1}{\Im_f},\cdots,\frac{z_k}{\Im_f})}{\frac{X^k}{\Im_{f_k}} \to \left(\frac{X}{\Im_f}\right)^k}$$

is bijective and $\mathrm{card}(\frac{X^k}{\Im_{f_k}}) = \left(\mathrm{card}(\frac{X}{\Im_f})\right)^k$.

(2) For $((x_1,\cdots,x_k),(y_1,\cdots,y_k)) \in \Re_{f_k}$ there exist $n,m \geq 1$ with $f_k^n(x_1,\cdots,x_k) = f_k^m(y_1,\cdots,y_k)$ thus for all $i \in \{1,...,k\}$ we have $f^n(x_i) = f^m(y_i)$ and $(x_i, y_i) \in \Re_f$ so

$$\underset{\frac{(z_1,\cdots,z_k)}{\Re_{f_k}} \mapsto (\frac{z_1}{\Re_f},\cdots,\frac{z_k}{\Re_f})}{\frac{X^k}{\Re_{f_k}} \to \left(\frac{X}{\Re_f}\right)^k}$$

is onto, thus $\mathrm{card}(\frac{X^k}{\Re_{f_k}}) \leq \left(\mathrm{card}(\frac{X}{\Re_f})\right)^k$, moreover

$$\underset{\frac{z}{\Re_f} \mapsto \frac{(z,\cdots,z)}{\Re_{f_k}}}{\frac{X}{\Re_f} \to \frac{X^k}{\Re_{f_k}}}$$

is one-to-one, hence $\mathrm{card}(\frac{X}{\Re_f}) \leq \mathrm{card}(\frac{X^k}{\Re_{f_k}})$.

Theorem 5. Consider nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{D_\tau : \tau > 1\}$ we have:

1. The following statements are equivalent:
   a. $h_2(\bigcup\mathrm{M}) \subseteq \bigcup\mathrm{M}$,
   b. one of the following conditions occurs:
   - $\mathrm{M} \cap (\{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\})$ is nonvoid,
   - $\mathrm{M} \cap \{D_\tau : \tau < \aleph_0\}$ is infinite,
   - $\mathrm{M} = \{D_2\}$,

   c. $D_{\aleph_0} \subseteq \bigcup\mathrm{M}$ or $\mathrm{M} = \{D_2\}$,
2. M is forwarding with respect to h_2 iff $h_2(\bigcup\mathrm{M}) \subseteq \bigcup\mathrm{M}$,
3. M is backwarding (resp. stationary) with respect to h_2 iff $\mathrm{M} \subseteq \{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\} \cup \{D_2\}$.

Proof. (1) (a) $\Rightarrow$ (b): Suppose $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$, $\mathrm{M} \cap (\{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\})$ is empty, and $\mathrm{M} \cap \{D_\tau : \tau < \aleph_0\}$ is finite, then there exist $n_1 < \cdots < n_s = p < \aleph_0$ with $\mathrm{M} = \{D_{n_j} : 1 \leq j \leq s\}$. So $\bigcup \mathrm{M} = D_p$, if $p > 2$ then consider $X = \{1, \ldots, p\}$ and $f : X \to X$ with $f(i) = i + 1$ for $i < p$ and $f(p) = p$, therefore

$$\frac{X}{\Im_f} = \{\{i\} : 1 \leq i \leq p-2\} \cup \{\{p-1, p\}\},$$

$\mathrm{card}(\frac{X}{\Im_f}) = p - 1 < p$ and $(X, f) \in D_p$. By Lemma 4(1),

$$\mathrm{card}(\frac{X^k}{\Im_{f_k}}) = (p-1)^k \geq 2(p-1) > p$$

and $h_2(X, f) = (X^k, f_k) \notin D_p = \bigcup \mathrm{M}$ which is in contradiction with $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$. Hence $n_s = 2$ and $\mathrm{M} = \{D_2\}$.

(b) $\Rightarrow$ (c): It's clear by definition of D_τ s.

(c) $\Rightarrow$ (a): Since for each transfinite cardinal number τ we have $\tau^k = \tau$ by Lemma 4(1) for each transfinite cardinal number τ we have $h_2(D_\tau) \subseteq D_{\tau^k} = D_\tau$ also for each $2 < n < \aleph_0$ we have $h_2(D_n) \subseteq D_{(n-1)^k+1} \subseteq D_{\aleph_0}$ moreover $h_2(D_2) \subseteq D_2$ which leads to the desired result.

(2) Use (1) and Lemma 4(1).

(3) First suppose $\mathrm{M} \subseteq \{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\} \cup \{D_2\}$, then by item (1), $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$. Using Lemma 4(1), M is backwarding and stationary with respect to h_2.

Now suppose M is backwarding with respect to h_2 and $\mathrm{M} \not\subseteq \{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq \aleph_0\} \cup \{D_2\}$. Then there exists finite $p > 2$ with $D_p \in \mathrm{M}$. Using the same method described in the proof of "(a) $\Rightarrow$ (b)" in item (1), there exists $(X, f) \in D_p$ with $h_2(X, f) \notin D_p$, which is a contradiction and completes the proof.

Theorem 6. Consider nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau > 1\}$ we have:

1. The following statements are equivalent:
   a. $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$,
   b. one of the following conditions occurs:
   - $\mathrm{M} \cap (\{\mathrm{SSet}\} \cup \{E_\tau : \tau \geq \aleph_0\})$ is nonvoid,
   - $\mathrm{M} \cap \{E_\tau : \tau < \aleph_0\}$ is infinite,
   - $\mathrm{M} = \{E_2\}$,

   c. $E_{\aleph_0} \subseteq \bigcup \mathrm{M}$ or $\mathrm{M} = \{E_2\}$,
2. M is forwarding with respect to h_2 iff $h_2(\bigcup \mathrm{M}) \subseteq \bigcup \mathrm{M}$,
3. M is backwarding (resp. stationary) with respect to h_2 iff $\mathrm{M} \subseteq \{\mathrm{SSet}\} \cup \{E_\tau : \tau \geq \aleph_0\} \cup \{E_2\}$.

Proof. For finite $p > 2$ consider $X = \{1, \ldots, p-1\}$ and identity map $f : X \underset{x \mapsto x}{\to} X$, then $\mathrm{card}(\frac{X}{\Re_f}) = p - 1$ and $(X, f) \in E_p$. However, $\mathrm{card}(\frac{X^k}{\Re_{f_k}}) = (p-1)^k \geq p$ and $h_2(X, f) \notin E_p$. Use Lemma 4(2) and a similar method described in the proof of Theorem 5 to complete the proof.

Note 7 (infinite self-product). For arbitrary infinite set Γ consider $h : \mathrm{SSet} \to \mathrm{SSet}$ with $h(X, f) = (X^\Gamma, f_\Gamma)$ with $f_\Gamma((x_i)_{i \in \Gamma}) = (f(x_i))_{i \in \Gamma}$. Then using similar method described in the finite case for each $(X, f) \in \mathrm{SSet}$ we have

$$\mathrm{card}(\frac{X^\Gamma}{\Im_{f_\Gamma}}) = \left(\mathrm{card}(\frac{X}{\Im_f})\right)^{\mathrm{card}(\Gamma)}$$

and

$$\mathrm{card}(\frac{X}{\Re_f}) \leq \mathrm{card}(\frac{X^\Gamma}{\Re_{f_\Gamma}}) \leq \left(\mathrm{card}(\frac{X}{\Re_f})\right)^{\mathrm{card}(\Gamma)}.$$

Thus for any nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{D_\tau : \tau > 1\}$ with $\mathrm{SSet} \in \mathrm{M}$, M is forwarding with respect to h. Also for nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{D_\tau : \tau \geq 2^{\mathrm{card}(\Gamma)}\}$, M is stationary with respect to h. Also for any nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau > 1\}$ with $\mathrm{SSet} \in \mathrm{M}$, M is forwarding with respect to h. Also for nonempty sub-class M of $\{\mathrm{SSet}\} \cup \{E_\tau : \tau \geq 2^{\mathrm{card}(\Gamma)}\}$, M is stationary with respect to h.

4. Third operator: disjoint union

Consider arbitrary set Γ with at least two elements and $h_3 : \mathrm{SSet} \to \mathrm{SSet}$ where $h_3(X, f) = (X \times \Gamma, f_{(\Gamma)})$ and $f_{(\Gamma)}(x, \gamma) = (f(x), \gamma)$ (as a matter of fact one may consider $h_3(X, f)$ "looks like" Γ copies disjoint union of (X, f)).

Lemma 8. For each $(X, f) \in \mathrm{SSet}$ we have:

$$\mathrm{card}(\frac{X \times \Gamma}{\Im_{f_{(\Gamma)}}}) = \mathrm{card}(\Gamma)\,\mathrm{card}(\frac{X}{\Im_f})$$

and

$$\mathrm{card}(\frac{X \times \Gamma}{\Re_{f_{(\Gamma)}}}) = \mathrm{card}(\Gamma)\,\mathrm{card}(\frac{X}{\Re_f}).$$

Proof. For each $(X, f) \in \mathrm{SSet}$ and $(x, i), (y, j) \in X \times \Gamma$ we have:

$$((x, i), (y, j)) \in \Im_{f_{(\Gamma)}} \Leftrightarrow (x, y) \in \Im_f \wedge i = j$$

and

$$((x, i), (y, j)) \in \Re_{f_{(\Gamma)}} \Leftrightarrow (x, y) \in \Re_f \wedge i = j.$$

Thus:

$$\begin{array}{c}\frac{X\times\Gamma}{\Im_{f(\Gamma)}}\to\frac{X}{\Im_f}\times\Gamma\\ \frac{(z,\gamma)}{\Im_{f(\Gamma)}}\mapsto(\frac{z}{\Im_f},\gamma)\end{array}$$

and

$$\begin{array}{c}\frac{X\times\Gamma}{\Re_{f(\Gamma)}}\to\frac{X}{\Re_f}\times\Gamma\\ \frac{(z,\gamma)}{\Im_{f(\Gamma)}}\mapsto(\frac{z}{\Im_f},\gamma)\end{array}$$

are bijective which lead to the desired result.

Theorem 9 (finite disjoint union). For finite Γ (with at least two elements) consider nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>1\}$ and nonempty sub-class M' of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau>1\}$, we have:

1. The following statements are equivalent:
   a. $h_3(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$,
   b. one of the following conditions occurs:
   - $\mathrm{M}\cap(\{\mathrm{SSet}\}\cup\{D_\tau:\tau\geq\aleph_0\})$ is nonvoid,
   - $\mathrm{M}\cap\{D_\tau:\tau<\aleph_0\}$ is infinite,

   c. $D_{\aleph_0}\subseteq\bigcup\mathrm{M}$,
2. M is forwarding with respect to h_3 iff $h_3(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$,
3. M is backwarding (resp. stationary) with respect to h_3 iff $\mathrm{M}\subseteq\{\mathrm{SSet}\}\cup\{D_\tau:\tau\geq\aleph_0\}$,
4. The following statements are equivalent:
   a. $h_3(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$,
   b. one of the following conditions occurs:
   - $\mathrm{M}'\cap(\{\mathrm{SSet}\}\cup\{E_\tau:\tau\geq\aleph_0\})$ is nonvoid,
   - $\mathrm{M}'\cap\{E_\tau:\tau<\aleph_0\}$ is infinite,

   c. $E_{\aleph_0}\subseteq\bigcup\mathrm{M}'$,
5. M' is forwarding with respect to h_3 iff $h_3(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$,
6. M' is backwarding (resp. stationary) with respect to h_3 iff $\mathrm{M}\subseteq\{\mathrm{SSet}\}\cup\{E_\tau:\tau\geq\aleph_0\}$.

Proof. For finite $p>1$ consider $X=\{1,\ldots,p-1\}$ and identity map $f:X\underset{x\mapsto x}{\to}X$ as in the proof of Theorem 6, then $\mathrm{card}(\frac{X}{\Im_f})=\mathrm{card}(\frac{X}{\Re_f})=p-1$ and $(X,f)\in E_p\cap D_p$. However

$$\mathrm{card}(\frac{X\times\Gamma}{\Im_{f(\Gamma)}})=\mathrm{card}(\frac{X\times\Gamma}{\Re_{f(\Gamma)}})=(p-1)\,\mathrm{card}(\Gamma)>p$$

and $h_3(X,f)\notin D_p\cup E_p$. Use Lemma 8 and a similar method described in Theorems 5 and 6 to complete the proof.

Note 10 (infinite disjoint union). For infinite Γ and nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>1\}$ with $\mathrm{SSet}\in\mathrm{M}$, M is forwarding with respect to h_3. Also for nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>\mathrm{card}(\Gamma)\}$, M is stationary with respect to h_3. Also for any nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau>1\}$ with $\mathrm{SSet}\in\mathrm{M}$, M is forwarding with respect to h_3. Also for nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau\geq\mathrm{card}(\Gamma)\}$, M is stationary with respect to h_3.

5. Fourth operator: induced map on power set

For arbitrary set X and cardinal numbers ϑ,θ let

$$\mathrm{P}^{<\vartheta}_{>\theta}(X)=\{A\subseteq X:\theta<\mathrm{card}(A)<\vartheta\}$$

$$\mathrm{P}^{<\vartheta}(X)=\{A\subseteq X:\mathrm{card}(A)<\vartheta\},$$

and $h_4:\mathrm{SSet}\to\mathrm{SSet}$ with $h_4(X,f)=(\mathrm{P}(X),\mathrm{P}(f))\in\mathrm{SSet}$ where $\mathrm{P}(f)(A)=f(A)(=\{f(x):x\in A\})$ (for $A\subseteq X$) also $h_4^{<\vartheta}(X,f)=(\mathrm{P}^{<\vartheta}(X),\mathrm{P}^{<\vartheta}(f))\in\mathrm{SSet}$ as the restriction of the above self-map to $\mathrm{P}^{<\vartheta}(X)$, i.e. $\mathrm{P}^{<\vartheta}(f)=\mathrm{P}(f)|_{\mathrm{P}^{<\vartheta}(X)}$.

Lemma 11. For $1<k<\aleph_0$ we have:

$$\mathrm{card}(\frac{X}{\Im_f})\leq\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})\leq\left(\mathrm{card}(\frac{X}{\Im_f})\right)^{2k-1}+1.$$

In particular for infinite $\frac{X}{\Im_f}$ we have $\mathrm{card}(\frac{X}{\Im_f})=\mathrm{card}(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}})$.

Proof. For each nonempty $A,B\in\mathrm{P}^{<k+1}(X)$ (i.e. $A,B\in\mathrm{P}^{<k+1}_{>0}(X)$) there exist $x_1,\ldots,x_k,y_1,\ldots,y_k\in X$ (may be not distinct) with $A=\{x_1,\ldots,x_k\},B=\{y_1,\ldots,y_k\}$. Now for $(X,f)\in\mathrm{SSet}$ and nonempty $A,B\in\mathrm{P}^{<k+1}(X)$ with $(A,B)\in\Im_{\mathrm{P}^{<k+1}(f)}$ and $A=\{x_1,\ldots,x_k\},B=\{y_1,\ldots,y_k\}$, we have

$$\mathrm{P}^{<k+1}(f)(A)=\mathrm{P}^{<k+1}(f)(B)$$

thus $\{f(x_1),\ldots,f(x_k)\}=\{f(y_1),\ldots,f(y_k)\}$, so for each $i\in\{1,\ldots,k\}$ there exist $s_i,t_i\in\{1,\ldots,k\}$ with $f(x_i)=f(y_{s_i})$ and $f(y_i)=f(x_{t_i})$. Without any loss of generality we may assume $f(x_1)=f(y_1)$ and $s_1=t_1=1$. Thus

$$(f(x_1),\cdots,f(x_k),f(x_{t_2}),\cdots,f(x_{t_k}))$$
$$=(f(y_{s_1}),\cdots,f(y_{s_k}),f(y_2),\cdots,f(y_k))$$

using the same notations as in the Second study we have $f_{2k-1}(x_1,\cdots,x_k,x_{t_2},\cdots,x_{t_k})=f_{2k-1}(y_{s_1},\cdots,y_{s_k},y_2,\cdots,y_k)$ and $((x_1,\cdots,x_k,x_{t_2},\cdots,x_{t_k}),(y_{s_1},\cdots,y_{s_k},y_2,\cdots,y_k))\in\Im_{f_{2k-1}}$ moreover, clearly we have

$$\{x_1,\cdots,x_k,x_{t_2},\cdots,x_{t_k}\}=\{x_1,\cdots,x_k\}$$

and $\{y_{s_1},\cdots,y_{s_k},y_2,\cdots,y_k\}=\{y_1,\cdots,y_k\}$. Hence the following map is onto

$$\left\{\frac{(z_i)_{1\leq i\leq 2k-1}}{\Im_{f_{2k-1}}}:\mathrm{card}\{z_1,\ldots,z_{2k-1}\}\leq k\right\}\to\frac{\mathrm{P}^{<k+1}_{>0}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}$$
$$\frac{(z_i)_{1\leq i\leq 2k-1}}{\Im_{f_{2k-1}}}\mapsto\frac{\{z_1,\ldots,z_{2k-1}\}}{\Im_{\mathrm{P}^{<k+1}(f)}}$$

(by $\frac{\mathrm{P}^{<k+1}_{>0}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}$ we mean $\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}$ except the equivalence class of empty set).

Therefore (use the Section 3 too):

$$\mathrm{card}\left(\frac{\mathrm{P}^{<k+1}_{>0}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}\right)\leq\mathrm{card}\left(\frac{X^{2k-1}}{\Im_{f_{2k-1}}}\right)=\left(\mathrm{card}\left(\frac{X}{\Im_f}\right)\right)^{2k-1}.$$

hence:

$$\mathrm{card}\left(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}\right)\leq\mathrm{card}\left(\frac{X^{2k-1}}{\Im_{f_{2k-1}}}\right)+1=\left(\mathrm{card}\left(\frac{X}{\Im_f}\right)\right)^{2k-1}+1$$

Moreover:

$$\frac{X}{\Im_f}\to\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}$$
$$\frac{z}{\Im_f}\mapsto\frac{\{z\}}{\Im_{\mathrm{P}^{<k+1}(f)}}$$

is one-to-one, thus

$$\mathrm{card}\left(\frac{X}{\Im_f}\right)\leq\mathrm{card}\left(\frac{\mathrm{P}^{<k+1}(X)}{\Im_{\mathrm{P}^{<k+1}(f)}}\right)\leq\left(\mathrm{card}\left(\frac{X}{\Im_f}\right)\right)^{2k-1}+1.$$

Corollary 12. For $1<k<\aleph_0$ we have:

$$\mathrm{card}\left(\frac{X}{\Re_f}\right)\leq\mathrm{card}\left(\frac{\mathrm{P}^{<k+1}(X)}{\Re_{\mathrm{P}^{<k+1}(f)}}\right)\leq\left(\mathrm{card}\left(\frac{X}{\Re_f}\right)\right)^{2k-1}+1.$$

Proof. Use a similar method described in Lemma 11.

Note 13. For $1<k<\aleph_0$, finite Γ (with at least two elements) nonempty sub-class M of $\{\mathrm{SSet}\}\cup\{D_\tau:\tau>1\}$ and nonempty sub-class M' of $\{\mathrm{SSet}\}\cup\{E_\tau:\tau>1\}$, we have:

- $h_3(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$ iff $h_4^{<k+1}(\bigcup\mathrm{M})\subseteq\bigcup\mathrm{M}$,
- M is forwarding (respectively backwarding, stationary) with respect to $h_4^{<k+1}$ iff it is forwarding (respectively backwarding, stationary) with respect to h_3,
- $h_3(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$ iff $h_4^{<k+1}(\bigcup\mathrm{M}')\subseteq\bigcup\mathrm{M}'$,
- M' is forwarding (respectively backwarding, stationary) with respect to $h_4^{<k+1}$ iff it is forwarding (respectively backwarding, stationary) with respect to h_3.

Proof. Use Theorem 9.

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Develop AES Algorithm based on Fuzzy Set Theory

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Abstract: Advance Encryption Standard AES cryptosystem is one of well-known block cipher that widely used to encrypt the sensitive data. However, attackers have pointed some drawbacks in the design of block ciphers, such as: (a) all block ciphers apply a same key for the encipherment of multiple blocks; (b) if adversary can discover the key for one block, he can immediately break the other blocks. Many security attacks have been applied on AES cipher including linear, differential, distinguishing, correlation and statistical attacks. The main objectives of this paper are; to develop a strong and high performance AES algorithm with the utilization of fuzzy function, to suggest three encryption approaches mixing AES with fuzzy function, and to analyze the security and evaluate the efficiency of developed algorithms. The result detects that the ciphertext acquired is the similar as the plaintext and fuzzy set theory was suitable for apply as round function in the design of other block ciphers. Moreover, the security properties, demonstrated that our designs were highly secure and robust against possible cryptographic attacks. Finally, the statistical test for randomness and comparison of the proposed ciphers with identical ciphers revealed that the proposed algorithms were efficient, and faster than the conventional block ciphers.

Keywords: Cryptography, AES, Fuzzy set theory, Security attacks, Statistical tests**.**

1. Introduction

Designers and attackers are always encoded in a constant competition to build new attackable codes; therefore, when broken, the new encryption proposal becomes necessary. For efficient coding of data, symmetric algorithms are used, in particular to block zeros through encryption. The researchers have confirmed the problems of mass zeros. It is said that all longitudinal zeros, for example, suffer from some typical weaknesses: (a) all spectral codes use one key to encode multiple blocks; (b) if the opponent can detect the key for one block, it can easily break the other blocks; Means that the opponent is able to collect many blocks encoded by one key which makes possible more attacks against one block. Many security attacks have been applied on AES such as differential, linear, distinguishing, correlation and statistical attacks.

Block cipher based on fuzzy set theory has become a rich research area in the field of computer security and cryptography. In the following, some of the published works in this area are reviewed.Madanayake, 2012 [1]. Proposed an algorithm provides security levels by using various keys based on fuzzy logic for the encryption / decryption process. Dhenakaran, S.S and.Kavinilavu, N, 2012 [2] Introduced a new method using a mysterious set theory to integrate text encryption and convert unclassified text from numerical to native using fuzzy logic. Hinal, et al.in 2015 [3] presented a cross-sectional approach based on logic technique using the secret sharing program (2, 2). Azam, N. A. 2017 [4] A new image encryption technique is recommended based on several AES Gray S (RTSs) technologies translated to the right. Abdullah, K. 2017 [5] Proposed a new RSA encryption system based on the theory of fuzzy set where the ciphertext and the plaintext are in terms of Triangular Fuzzy Number (TFN). In order to bridge the gaps in AES algorithm, and because of need arises for guarantee the security of the block cipher cryptosystems while the communication must be ensured, it is a good idea to developed AES block cipher algorithm based on fuzzy set theory whereby the plaintext and the ciphertext are in terms of Triangular Fuzzy Number (TFN). The rest of the paper as in the following; in section 2, describe briefly AES algorithm, while some type of fuzzy set functions was discussion in section 3, in section 4, the suggested algorithms with some their properties was illustration, finally, results and discussion, conclusions and further works in sections 5, 6 and 7 respectively.

2. AES Algorithm

The National Institute of Standards and Technology (NIST) began the search for an alternative to the Data Encryption Standard (DES). In 2002 [6] 1997. the Advanced Encryption Standard (AES) is the new standard, as shown in Figure 1, developed by Joan Damen and Vincent Regman. It encrypts/decrypts data in 128-bit clusters using 128-bit (10-rounds), 192-bit (for 12 rounds) and 256-bit (14-round) key sizes; each round includes stages different processing consists of substitution, conversion, mixing of ordinary text of income, and conversion to the final output of encoded text. It is more secure than DES and 3DES, moreover, it is general design, flexible and available worldwide for free [6].

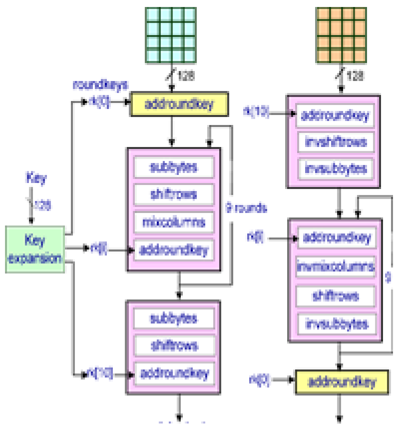

Figure 1. Shows AES algorithm [6].

All operations of AES is applied in GF(2^8) and the sixteen bytes of the 128-bit input block a0,…,a15 being arranged in a (4 × 4) matrix of bytes as shown in Figure 2.

a_0	a_4	a_8	a_{12}
a_1	a_5	a_9	a_{13}
a_2	a_6	a_{10}	a_{14}
a_3	a_7	a_{11}	a_{15}

Figure 2. Shows the (4 × 4) matrix of bytes for AES

Each round uses four actions, as shown in Figure 3, named "ShiftRows","SubBytes","MixColumns"and"AddRoundKey" . The last round has a slightly different shape and deletes the MixColumns process. The encryption begins with the AddRoundKey process, and then, the SubBytes process, at which point each byte is replaced by a byte of a reversible S-box. In the ShiftRows process, the rows (for bytes) are converted to a number of byte locations to the left; the first row is not shifted, the second row is shifted through one position, the third row to two, and the last row three. The last process is MixColumns. At this stage, the four bytes in each column are mixed by the quadrature of the four-byte vector by a constant, reversible, (4×4)-matrix over GF(2^8). The main characteristic is that if two types of input vectors are different in bytes s, the output variables differ in at least5 - s bytes, where 1 ≤ s ≤ 4. Each round closures with AddRoundKey, where 16 round-key bytes are xor'ed to the 16 information bytes. AES has generally straight forward key calendar of length16, 24, and 32 bytes, this key was expanded and returns of 16 × 11, 16 × 13, and 16 × 15 bytes respectively [7].
Decipherment applied the inverse process of encipherment, therefore, when we used the same key at encryption then the plaintext will be receives in decryption.

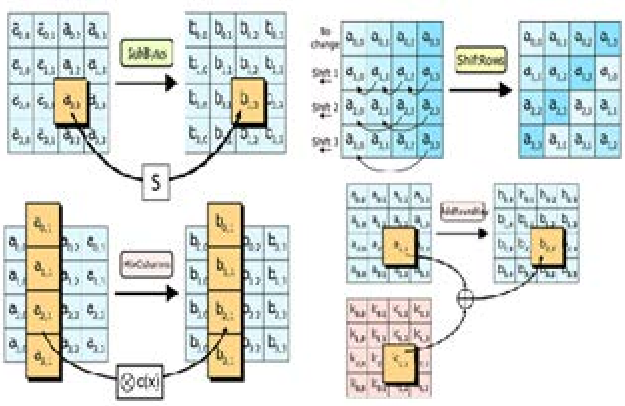

Figure 3. Shows the four transformations of AES algorithm

3. Fuzzy Sets and Fuzzy Membership Functions

Normally, an object has a numeric (degree of membership) between 0 and 1, 0 membership means the object is not in the set,1 membership means the object is fully inside the set and in between means the object is partially in the set. the description of this fact in mathematic can be represented as, If U is a collection of objects denoted generically by x, then a fuzzy set A in U is can be defined as a set of ordered pairs: $A = \{(x, \mu_A(x)) | x \in U\}$, where U : universe of discourse, and $\mu_A: U \to [0,1]$. Characteristic function μ, indicating the belongingingness of x to the set A,

$$\mu_A(x) = \begin{cases} 1 \; if \; x \in A \\ 0 \; if \; x \notin A \end{cases}$$

called membership. The membership functions that characterize the blurry groups and the assemblies used are the basis of fuzzy sets and fuzzy logical systems [8].
Fuzzy logic (FL) is a control system (or logical) of the n-logic system that uses the "or fact" of the inputs and produces outputs based on the input states and the rate of change (instead of the normal "error or error" (1 or 0) , and the logic of low or high (binary) depends on the basis of the modern computer, it provides the basis for the approximate thinking using inaccurate decisions and allows the use of linguistic variables uses FL as a mathematical tool in areas such as job optimization, filtration and installation curves, etc. [9].
The Fuzzy Logic application itself to a special system is in fact not very different from applying logical logic or probability logic. The FL difference comes from its ability to create a more general theory of the decision-making process, called the foggy processor, a special case of approximate inference. The hazy wizard uses a blurry set and FL theory in the logical thinking process and acts as a vague logic algorithm. Ambiguous logic or is made through the mysterious words that we use so much in our daily lives. For example, expressions like a little [10].
Fuzzy membership functions can be seen as a bridge between uncertain data and a blurry world. Organic functions representing mysterious groups have different forms, which are determined by certain types of mathematical formulas. The most common types of functions include trigonometric, trapezoidal, triangular, bell, sinusoid, Gaussian, Cauchy and sigmoid. In order to make operations on cloud groups easier, membership functions are formulated according to their parameters, which include information about the ambiguity and scope of the site in the discourse world. Flexibility in parameter settings makes membership functions also adjustable. Because of the linearity of its structure, it is preferable to use organic functions of the triangular type over others [11].
Some properties of Triangular Membership Functions (TMF) are briefly examining in the following subsection.

3.1 Triangular Membership Function (TMF)

Triangular membership functions can made of lines, as shown in Figure 4, and realized by the combination of line equations given in:

$$M_A(x) = \begin{cases} 0 & if\ x > x_1 \\ \frac{x-x_1}{x_2-x_1} & x_1 \le x \le x_2 \\ \frac{x_3-x}{x_3-x_2} & x_2 \le x \le x_3 \\ 0 & if\ x > x_3 \end{cases} \qquad (1)$$

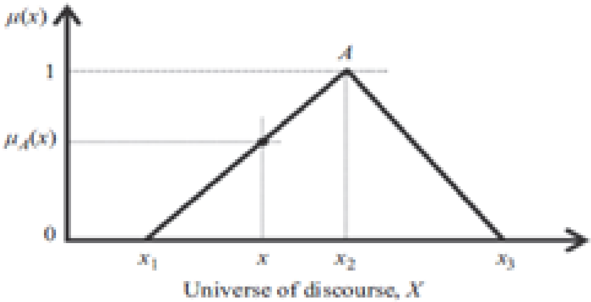

Figure 4. Shows Triangular fuzzy membership function

Where the parameters x1, x2 and x3 give the location of fuzzy membership function A in the X universe as shown in Figure 4. In fact, the parameters x1, x2, and x3 represent the function of membership A and show us its location in the opposite universe. It is sufficient to change parameter values in order to determine a new membership function of a similar format or to change the location of the speech. This is why the parameter formulas are important for representing membership functions. Relation (1) can be used as a parameterized membership function that represents ambiguous subsets of the triangular type. Equation (1) shows that x2 is a convergence point and equation (2):

$$M_A(x) = \left(\frac{x-x_1}{x_2-x_1}\right) < \left(\frac{x_3-x}{x_3-x_2}\right) \qquad (2)$$

can be satisfied as long as x1 ≤ x2 and x ≤ x2.

$$M_A(x) = \left(\frac{x3-x}{x3-x2}\right) < \left(\frac{x-x1}{x2-x1}\right) \qquad (3)$$

Similarly, equation (3) is satisfied as long as x ≥x2 and x2 ≤ x3. In other words, the output is equal to the smaller part of (2) or (3). However, these equations give a negative output if x < x1 or x > x3. Since the membership scores are set at a time interval [0,1], negative outputs must be changed to 0. Therefore, the maximum value must be set between 0 and output from (2) or (3). Accordingly, (1) can be converted to the figure in (4):

$$M_A(x) = max\left(min\left(\frac{x-x1}{x2-x1}, \frac{x3-x}{x3-x2}\right)\right) \qquad (4)$$

Triangular fuzzy subsets are simple to model and very easy to simulate. The sharp peak can them to react to any changes even if they are very small. Thus, sharp peak produces triangle membership functions critical to the changes in the fragile variable x [12].

4. Fuzzy-AES algorithms

This section provided the mathematical basis of proposed algorithms, some of the planners used to structure modern restore block ciphers and modes of procedure. Further, it represents the design of a new efficient and secure block cipher called Fuzzy-AES algorithm. Two major parts are producing by the proposed algorithms: Fuzz set theory and AES algorithm, which is used to implement the encipherment and decipherment processes. From this situation, the name comes Fuzzy-AES. At the beginning of this section, a focus on investigating the structure of Fuzzy-AES algorithms, types of Fuzzy-AES algorithms with encryption/decryption processes. Finally, the significant properties and some advantages to justify the correctness of the proposed algorithms are discussing. The methodology for Fuzzy-AES is demonstrated in Figure 5.

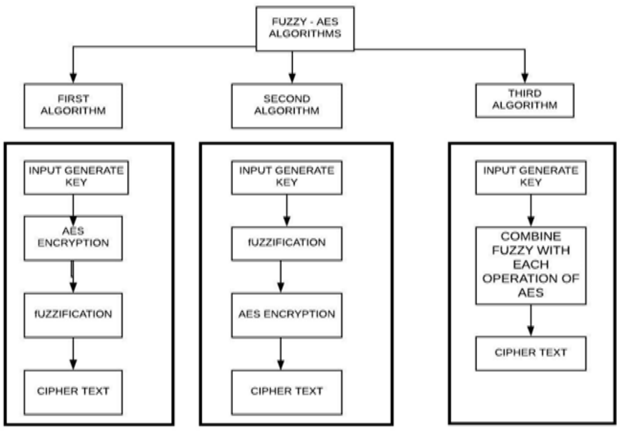

Figure 5. Block diagram of Fuzzy-AES algorithms

As illustration in Figure 5, Fuzzy-AES algorithm consists of the following parts:

a) First Algorithm: start with AES algorithm then used Fuzzy function.
b) Second Algorithm: start with Fuzzy function then used AES algorithm.
c) Third Algorithm: combine between Fuzzy function with each operations of AES algorithm.

The inputs of these algorithms are:

1. The key (K): it is the master keystream of the Fuzzy-AES algorithms, which generated from pseudorandom Number Generator PRNG, applying Cipher Block Chaining CBC mode of operation. It consists of 16-bytes $(k_0, k_1, k_2, \dots, k_{15})$, that is input into Fuzzy-AES algorithms to generate the new ciphertext for each round.
2. The plaintext $P1, P2, P3, P4, \dots$: it is the message required to encode. The plaintext P is comprised by

16-bytes $(p_0, p_1, \dots, p_{15})$.

While the output of these algorithms is the Ciphertext (C_i), which was the final output of algorithm, where the ciphertext consists of $C1, C2, C3, C4, \dots$ with each C comprised of 16-bytes $(c_0, c_1, \dots, c_{15})$ for each round, it is returning to PRNG for generated a new keystream.

The following subsections deal with various algorithms with theirs properties.

4.1 First algorithm

In this algorithm, first applied the same procedure operations of AES algorithm to the input data, then make fuzzicaition for the output data by employing a Triangular Fuzzy Membership (TFM) map to produce the ciphertext of this algorithm, as shown in Figure 6.

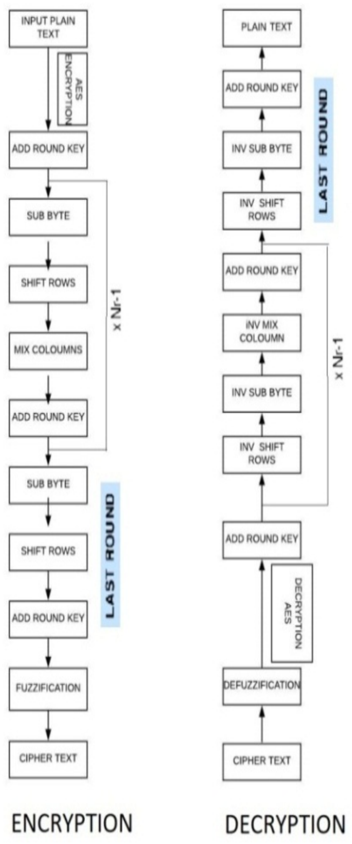

Figure 6. Encryption/Decryption process of 1st algorithm

4.2 Second algorithm

The second algorithm begin with fuzzication the input by applying a triangular fuzzy membership map, then applied the same procedure operation of AES algorithm for the output to produce the ciphertext of this algorithm as shown in the Figure 7.

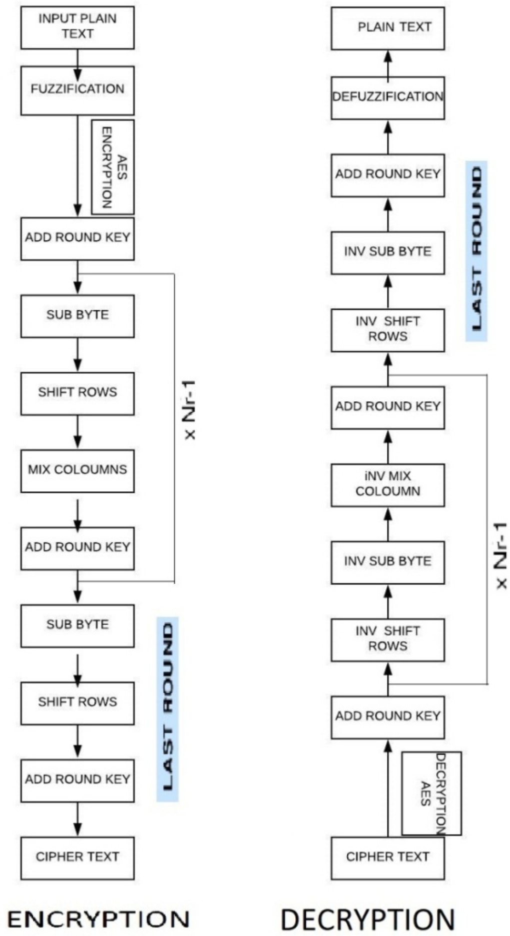

Figure7. Encryption/Decryption process of 2nd algorithm

4.3 Third algorithm

In the third algorithm, start with AES algorithm that has four stages, namely AddRound key, SubByte, ShiftRow and MixColumn, and for each round, applied these operations as the following steps:

a) Applied AddRound key stage, which represented the first operation of AES to the input, then make fuzzicition to the output of this stage by using a triangular fuzzy membership map, as describe in subsection 3.1.
b) Next, the result input to SubByet stage, which represented the second operation of AES, then make fuzzicition to the output of this stage by using TFM.
c) After that, the result input to the ShifRow stage, which represented the third stage of AES, then make fuzzicition to the output of this stage by using TFM.
d) Finally, the result input to MixColumn stage, which represented the final stage of AES, then make fuzzicition to the output of this stage by using TFM.
e) The final output represented the ciphertext of this algorithm as shown in Figure 8.

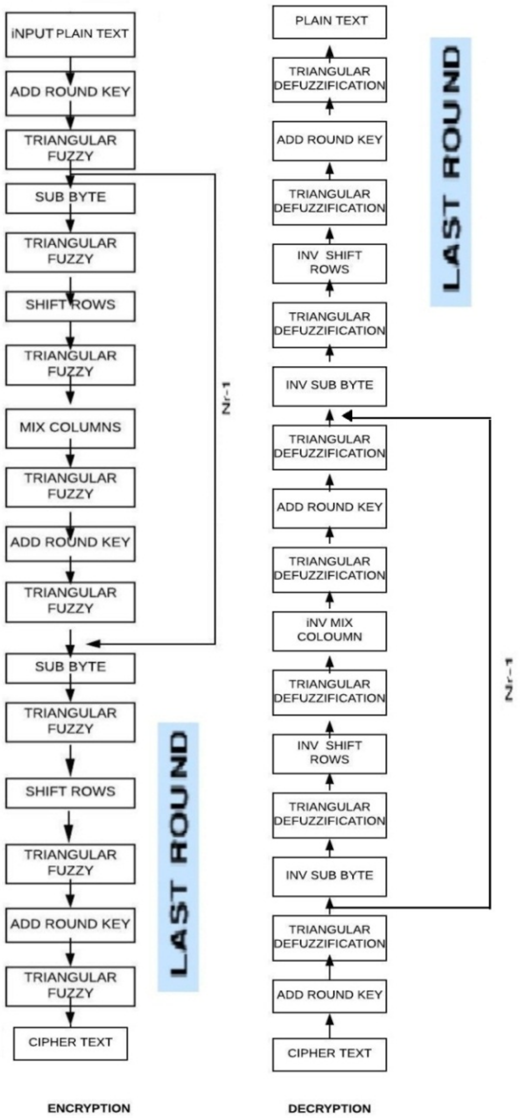

Figure7. Encryption/Decryption process of 3rd algorithm

4.4 Properties of Fuzzy-AES algorithms

1. Fuzzy-AES algorithm was characterized by its ability to produce greater security with proper implementation and by generating new functions between the cluster cryptography round and the fuzzy set theory.
2. In these new designs, a fuzzy set theory was proposed to produce an effective and long cyst. Some randomized sequences, which cannot be distinguished from truly random sequences, can be used for cryptographic system applications. These semi-random sequences use a larger number of alphabets for these purposes to increase the number of possibilities.
3. The main purpose of designing Fuzzy-AES algorithms is to use appropriate and effective PRNG along with the appropriate uniform.
4. Thus, we can realize that Fuzzy-AES algorithms, as more efficient and powerful algorithms, have an additional positive effect on plaintext and keystream. They have the ability to generate a kind of balance in their structure.
5. Moreover, they can produce a real encoder resulting from the plain text mixture and the keystream. As a result, fuzzy set theory PRNG is deploying by the Fuzzy-AES to generate the keystream, which is feature by high-level security and performance.
6. Occasionally, a triangular fuzzy membership map as a function with special properties, working on the two inputs (i.e., plaintext and keystream), has the ability to return a copy of the encrypted text to PRNG, which generates the next keystream.
7. As a result, the PRNG-encoded comment process results in a greater advantage for Fuzzy-AES than the three-pointed post function.
8. There is a similarity in design between the proposed algorithms and other ciphers. Fuzzy-AES is a self-synchronous block encoding where encrypted text has an effect on the image key. It has a high level of safety.
9. Intentional fuzzy-AES algorithms are intending for use with 16-byte keystream. This Keystream is using in PRNG to produce a new cyst of up to 64 bytes. In each round of AES-Fuzzy algorithms, PRNG generates a 16-byte keystream by combining, using Nonlinear Invertible Round Function (NLIRF), with 16-byte plaintext to generate 16-byte encrypted text.

5. Results and discussions

This section addresses the major issues regarding Fuzzy-AES algorithms; it examines the performance of these algorithms along with possible security attacks and administers the binary digits randomness tests of the ciphertext bits for these algorithms. A brief survey of the security analysis for Fuzzy-AES algorithms is providing in the following subsections.

5.1 Possible Attacks against Fuzzy-AES algorithms

a) **Brute-force Attack:** In this type of attack, the adversary try all possibilities**.**Since Fuzzy-AES algorithms applied 128-bit as a keystream, therefore the attacker needs 2^{128} possible keys, which approximately equal 3.4×10^{38} keys, this mean that the time required at one encryption per μs was approximately equal to $2^{127} = 1.7 \times 10^{38}$ years in order to apply a brute force attack against Fuzzy-AES algorithms [13]. Hence, an exhaustive key search attack took a long time and it appears infeasible.

b) **Ciphertext Only Attack:** The adversary has only a number of ciphertext messages and tries to discovery any relationships between the ciphertext and the data that expose the cipher system till the ciphertext message is solved. In Fuzzy-AES algorithms, the plaintext data that input to fuzzy set theory is randomized and Perform with a keystream sequence through an exclusive or operation. Then, the result data can be changed through many conversion stages of the round function that include the byte and transformation rows Mixcolumn and AddRound Key. As a result, the statistical properties of the plain text message will be removed and the resulting encrypted text message will result in nearrandomization. Thus, the attack appears to be encrypted text is not possible.

c) **Known Plaintext Attack:** The adversary in this type of attack need the plaintext corresponding the ciphertext.The plaintext bytes in Fuzzy-AES algorithms are XOR'ed with the keystream bytes, the resulting bytes are substitute by employing AES transformation and fuzzy function. The secret key that used to update AES transformation and fuzzy function make the opponent unable to determine the plaintext byte. Therefore, it was difficult to applied known plaintext attack.

d) **Statistical Attack:** tests of statistical are been performed on Fuzzy-AES algorithms, e.g. Frequency test, Serial test, Poker test, Runs test and Auto-correlation test [14]. In the current AES-Fuzzy algorithms, keystream and ciphertext have been adopted on the mysterious group theory functions and the AES algorithm to produce effective cryptographic text. As a result, to ensure that the new encrypted text remains strong, the bits of encoded text in the proposed algorithms have been tested extensively with the application of statistical tests of different lengths. The resulting encoded output passed all statistical tests, including randomized, binary numbers (see Tables 1, 2, 3, and 4) that justified the generation of encrypted text.

e) **Differential Analysis Attack:** Differential Analysis Attack is a generic term for all kind of cryptanalysis which investigates how differences in the information input can result in differences in the output. This attack seems undetectable to apply on Fuzzy-AES algorithms since the S-box is update by secret key for each round. Accordingly, the opponent does not have any information about the arrangement of S-box.

f) **Distinguishing and Correlation Attacks:** Two sets of attacks (i.e. differential and correlational) which closely resemble each other are discussing in this part. Any type of cryptanalysis which is applied in order to distinguish the encoded data from random data is called by the generic term distinguishing analysis or attack. Correlation analysis refers to a class of known plaintext attacks, employing Boolean function. A weakness in the choice of Fuzzy-AES algorithm Make encryption functionality susceptible to link analysis. It is recommended to choose a logical function that cannot be exploited by correlation analysis. In general, designers should exercise caution when applying zeros using the logical function.

5.2 Basic Five Binary Digits Statistical Tests

Random property and ciphertext bits are analyzed by applying the five statistical tests named Frequency test, Serial test, Poker test, Runs test and Auto-correlation test [15]. The frequency test is for uniformity and the other tests are for independence. These tests are a fundamental package that is usually applying for block cipher, stream cipher and keystream generation [16]. The resulting values of each test were comparing with the corresponding value of Chi square distribution. The keystream and ciphertext generated in the proposed Fuzzy-AES algorithms for different key length sizes are successfully passed all these tests for every run. A summary of the results are giving in Tables 1, 2, 3 and 4.

Table 1. Statistical tests for the master keystream of the Fuzzy-AES algorithms

Tests	***1st alg =128bit***	***2nd alg =128bit***	***3rd alg =128bit***	***Pass value***	***Result***
Frequency	0.654	2.793	0.0312	≤ 3.841	Pass
Serial test	-26.088	-31.599	0.0118	≤ 5.991	Pass
Poker test	-32.000	-8.000	6.761	≤ 14.067	Pass
Run test	3.559	3.873	1.882	≤ 22.362	Pass
Auto correlation					
Shift 1	-28.751	-3.882	0.266	≤ 1.960	Pass
Shift 2	-12.644	-14.462	0.178		Pass
Shift 3	-18.814	-9.458	-1.162		Pass
Shift 4	-19.894	-1.232	1.616		Pass
Shift 5	-18.064	-3.639	-0.631		Pass
Shift 6	-14.910	-12.631	-0.543		Pass
Shift 7	-17.667	-1.044	0.272		Pass
Shift 8	0.602	-5.595	-1.278		Pass
Shift 9	-8.294	-6.411	-1.191		Pass
Shift 10	-21.865	-1.5807	1.104		pass

Table 2. Statistical tests for ciphertext of **1st** algorithm with different key lengths

Tests	***Key length =128bit***	***Key length =512bit***	***Key length =1024 bit***	***Pass value***	***Result***
Frequency	1.3333	0.363	0.568	≤ 3.841	Pass
Serial test	4.1273	2.531	4.272	≤ 5.991	Pass
Poker test	-16.000	-4.000	-64.000	≤ 14.067	Pass
Run test	1.1990	1.022	1.402	≤ 22.362	Pass
Auto correlation				≤ 1.960	
Shift 1	-0.619	-2.110	-16.879		Pass
Shift 2	-7.818	-20.009	-9.961		Pass
Shift 3	-6.469	-0.311	-11.094		Pass
Shift 4	-7.964	-18.819	-3.934		Pass
Shift 5	0.583	-0.230	-9.016		Pass
Shift 6	-1.368	-16.111	-17.368		Pass
Shift 7	-0.739	-3.818	-11.918		Pass
Shift 8	-4.231	-14.677	-0.454		Pass
Shift 9	-8.485	-12.274	-14.454		Pass
Shift 10	-9.200	-11.018	-3.689		Pass

Table 3. Statistical tests for ciphertext of **2nd** algorithm with different key lengths

Tests	*Key length =128bit*	*Key Length =512 bit*	*Key length =1024 bit*	*Pass value*	*Result*
Frequency	0.278	0.187	0.000	≤ 3.841	Pass
Serial test	3.424	2.969	1.900	≤ 5.991	Pass
Poker test	-2.000	-4.000	-16.000	≤ 14.067	Pass
Run test	1.287	1.235	1.020	≤ 22.362	Pass
Auto correlation					
Shift 1	-13.947	-6.585	-6.092	≤ 1.960	Pass
Shift 2	-3.863	-5.810	-1.772		Pass
Shift 3	1.734	-0.309	-7.363		Pass
Shift 4	-3.863	-4.193	-0.889		Pass
Shift 5	-15.758	-8.547	0.000		Pass
Shift 6	-7.191	-9.293	-1.285		Pass
Shift 7	-4.483	-8.040	0.315		Pass
Shift 8	-4.134	-4.209	-4.055		Pass
Shift 9	-2.629	-1.728	-4.647		Pass
Shift 10	-5.973	-2.611	-2.607		Pass

Table 4. Statistical tests for ciphertext of 3rd algorithm with different key lengths

Tests	*Key length =128bit*	*Key Length =512 bit*	*Key length =1024 bit*	*Pass value*	*Result*
Frequency	1.531	2.000	3.781	≤ 3.841	Pass
Serial test	1.767	2.244	4.809	≤ 5.991	Pass
Poker test	-2.000	-4.000	-15.500	≤ 14.067	Pass
Run test	0.602	-0.253	0.887	≤ 22.362	Pass
Auto correlation				≤ 1.960	
Shift 1	-1.400	-3.400	-1.069		Pass
Shift 2	-2.155	-1.708	-6.075		Pass
Shift 3	-1.448	-0.545	-7.379		Pass
Shift 4	-1.454	-2.412	-5.488		Pass
Shift 5	-4.989	-2.233	-6.974		Pass
Shift 6	-3.713	-4.314	-2.435		Pass
Shift 7	-2.008	-6.708	-0.736		Pass
Shift 8	-5.031	-0.360	-4.800		Pass
Shift 9	-6.037	-4.663	-2.967		Pass
Shift 10	0.381	-7.293	-4.800		Pass

Moreover, the statistical tests compression for 1st , 2nd and 3rd algorithms for different key length sizes are successfully passed all five statistical tests for every run. A summary of the results is presenting in Tables 5, 6 and 7.

Table 5. Statistical tests compression of 1st , 2nd and 3rd algorithms for key length 128-bit

Tests	*1st alg =128bit*	*2nd alg =128bit*	*3rd alg =128bit*	*Pass value*	*Result*
Frequency	1.333	2.793	2.531	≤ 3.841	Pass
Serial test	4.127	-31.599	3.224	≤ 5.991	Pass
Poker test	-16.000	-8.000	-42.000	≤ 14.067	Pass
Run test	1.199	3.873	0.033	≤ 22.362	Pass
Auto correlation				≤ 1.960	
Shift 1	-0.619	-3.882	-1.546		Pass
Shift 2	-7.818	-14.462	-5.031		Pass
Shift 3	-6.469	-9.458	-4.451		Pass
Shift 4	-7.964	-1.232	-3.863		Pass
Shift 5	0.583	-3.639	0.000		Pass
Shift 6	-1.368	-12.631	-.122		Pass
Shift 7	-0.739	-1.044	-7.672		Pass
Shift 8	-4.231	-5.595	-6.841		Pass
Shift 9	-8.485	-6.411	-1.100		Pass
Shift 10	-9.200	-1.5807	-2.525		pass

Table 6. Statistical tests compression of 1st , 2nd and 3rd algorithms for key length 512-bits

Tests	*1st alg =512bit*	*2nd alg =512bit*	*3rd alg =512bit*	*Pass value*	*Result*
Frequency	0.363	2.793	0.187	≤ 3.841	Pass
Serial test	2.531	-31.599	2.969	≤ 5.991	Pass
Poker test	-4.000	-8.000	-4.000	≤ 14.067	Pass
Run test	1.022	3.873	1.235	≤ 22.362	Pass
Auto correlation				≤ 1.960	
Shift 1	-2.110	-3.882	-6.585		Pass
Shift 2	-20.009	-14.462	-5.810		Pass
Shift 3	-0.311	-9.458	-0.309		Pass
Shift 4	-18.819	-1.232	-4.193		Pass
Shift 5	-0.230	-3.639	-8.547		Pass
Shift 6	-16.111	-12.631	-9.293		Pass
Shift 7	-3.818	-1.044	-8.040		Pass
Shift 8	-14.677	-5.595	-4.209		Pass
Shift 9	-12.274	-6.411	-1.728		Pass
Shift 10	-11.018	-1.5807	-2.611		pass

Table 7. Statistical tests compression of **1st , 2nd** and **3rd** algorithms for key length 1024-bits

Tests	*1st alg =1024bit*	*2nd alg =1024*	*3rd alg =1024*	*Pass value*	*Result*
Frequency	0.568	0.000	3.781	≤ 3.841	Pass
Serial test	4.272	1.900	4.809	≤ 5.991	Pass
Poker test	-64.000	-16.000	-15.500	≤ 14.067	Pass
Run test	1.402	1.020	0.887	≤ 22.362	Pass
Auto correlation				≤ 1.960	
Shift 1	-16.879	-6.092	-1.069		Pass
Shift 2	-9.961	-1.772	-6.075		Pass
Shift 3	-11.094	-7.363	-7.379		Pass
Shift 4	-3.934	-0.889	-5.489		Pass
Shift 5	-9.016	0.000	-6.974		Pass
Shift 6	-17.368	-1.285	-2.435		Pass
Shift 7	-11.918	0.315	-0.736		Pass
Shift 8	-0.454	-4.055	-4.800		Pass
Shift 9	-14.454	-4.647	-2.967		Pass
Shift 10	-3.689	-2.607	-4.880		pass

Finally, the comparison for 1st, 2nd and,3rd algorithms with identical cipher algorithms such as AES, DES and 3DES has been done . A summary of the results is available in Table 8.

Table 8. Compression between **1st , 2nd** and **3rd** algorithms and Identical Algorithms [17]

Algorithms	*Key Length*	*No.of Round*	*Block Size*	*Cryptanalysis Resistance*	*Security Level*	*Possible Keys*	*Time Required*
DES	**65 bits**	**16**	**64 bits**	**Vulnerable to Differential & linear attacks**	**Proven inadequate**	2^{56}	**For 56 bits key =400 days**
3DES	**K1 , k2, k3=168 bits**	**48**	**64 bits**	**Vulnerable to Brute Force, plain text & differential attacks**	**One only weak which is exit in DES**	2^{112} 2^{168}	**For 112 bits =800 days**
AES	**128 192 256**	**10 12 14**	**128 192 256**	**Absolut Strong against Differential, Linear , interpolation & square attack**	**Considered secure**	2^{128} 2^{192} 2^{256}	**For 128 bit key =5*10^{21} years**
1st alg.	**128**	**10**	**128**	**Strong against Differential , linear , statistical attack**	**Considered secure**	2^{128}	**For 128 bit key =5*10^{21} years**
2st alg.	**128**	**10**	**128**	**Strong against Differential , linear , statistical attack**	**Considered secure**	2^{128}	**For 128 bit key =5*10^{21} years**
3st alg.	**128**	**10**	**128**	**Strong against Differential , linear , statistical attack**	**Considered secure**	2^{128}	**For 128 bit key =5*10^{21} years**

6. Conclusions

The current paper attempt to discuss the possibilities of developing a new block cipher algorithms of more efficiency (pass the statistical tests for randomness) and security (resists against security attacks) than other block ciphers. In this paper, the mechanism used to develop the weak classical concept of the AES algorithm worked to form a stronger and more suitable coding. A little later, this paper attempted to introduce new block encryption algorithms called Fuzzy-AES. Thorough tests have been done by describing these algorithms, evaluating their performance and security properties, and examining their implementation aspects. The analysis of the devised algorithms demonstrated that the proposed algorithms are characterized by flexibility; speed; sufficient; and highly secured than similar block ciphers such as DES, 3DES and AES. In the future, NIST tests can be used to show a promising building block for cryptographic systems with certain advantages over ambiguous set theorems and fuzzy logic. With a new blur mechanism by applying an additional type of organic functions such as Gaussian, Cauchy and Bell

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Behavior of Visible Submodules in the Class of Multiplication Modules

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Abstract

In this study , we suppose that T is a commutative ring with identity and X is a unitary module on T. A proper submodule W of a module X over a ring T is called visible if for every nonzero ideal I of T , implies $W = IW$ where this concept is up (to our knowledge). Here the behavior of the above concept has been studied within the class of multiplication modules. Some of the distinctive results has been submitted also,the trace of visible submodule has been presented where it was symbolized by $Tr(W)$. Two important descriptions for the trace of visible submodule of multiplication module have been given , also we have demonstrated when the visible submodule of multiplication are torsionless , add to that many properties of $Tr(W)$ have been reviewed.

Keywords

Visible submodule , divisible module , multiplication module , cancellation module , torsionless module , flat module.

Introduction :

T stands for commutative ring with identity and X for the unitary module over T. In [1] , Buthyna N. Shihab and Mahmood S. Fiadh submitted a concept of visible submodule which is defined as a proper submodule W of a module X over a ring T , so that it achieves $W = IW$ for every nonzero ideal I of T. Many of the properties which characterize this concept have been built add to a lot of important results and features have been submitted in [1]. Also , Buthyna N. Shihab and Mahmood S. Fiadh are given in [2] the concept of fully visible module where the module X on T is called fully visible if each submodules of it is visible. The properties and characteristics of this concept have also been reviewed in addition to other results. The aim of this article is to look for the behavior and effectiveness of the visible submodules within the class of multiplication modules. Where many properties have been proved and other important outcomes have been incorporated that adopt the same relationship. In addition to this , the trace of visible submodule has been provided. Two descriptions for the trace of visible submodule which are encoded by $Tr(W)$ have been mentioned. Finally we discussed the conditions under which the trace of visible submodule of multiplication module is torsionless. In our article we will need a number of basic concepts that we will include here.

- An T-module X is called multiplication if $\forall W \leq X$ (W submodule of) , $\exists$ ideal I of Tsuch that $W = IX$ [3].
- Let K , W be two T-submodules of X. Then the residual of K by W is the set of all $s \in T$ such that $sW \sqsubseteq K$ and dented by $(K: W)$. The annihilator of X is written as $(0: X)$ and dented by $ann_T(X)$, if $ann_T(X)$ is equal to zero , then X is said to be faithful [4].
- A submodule W of an T-module X is named multiplication submodule of $X \Leftrightarrow W \cap K = (W: K)K$ for every submodule K of X [5].
- An idempotent submodule W of a module X over T is defined as : W is an idempotent $\Leftrightarrow W = (W: X)W$ [6].
- An T-module X is called cancellation module if $IX = JX$ for any two ideals I and J of , implies $I = J$ [7].
- An T-module X is called fully cancellation module if for each ideal I of T and for each submodules N_1 , N_2 of X such that $IN_1 = IN_2$ implies $N_1 = N_2$ [8].
- An T-module X is flat if for each injective homomorphism $f: N' \longrightarrow N$ from one T-module to another , the homomorphism $I_X \otimes_T f: X \otimes_T N' \longrightarrow X \otimes_T N$ is injective , where I_X is the identity isomorphism of X [6].
- An T-module X is called divisible if and only if $rX = X$ for each $0 \neq r \in T$ [9].

1.Visible submodule of multiplication module

A proper submodule W of an T- module X is said to be visible , if $W = IW$ for every nonzero ideal I of T. In this part , the behavior of visible submodule was studied in the class of multiplication module where a distinction was given to the submodule because we proved that $W \leq X$ is visible if and only if $(W: X)$ is visible ideal of T when X is multiplication faithful finitely generated module. Many properties and useful results have introduced.

Under the class of multiplication and cancellation module , we have the following characterization.

Proposition(1.1):

Let X be a multiplication cancellation T-moduel. Then every proper submoduel N of X is visible submoduel if and only if $(N:X)$ is visible ideal of T.

Proof:

$\Leftarrow$) Suppose that $(N:X)$ is visible ideal of X. Let $x \in N$. Then $(x) \sqsubseteq N$ and hence $((x):_{T} X) \sqsubseteq (N:_{T}X)$.

Therefore $((x):_{T} X) \sqsubseteq (N:_{T}X) = I(N:X)$.

hence $((x):_{T} X)X \sqsubseteq I(N:_{T}X)X$ which implies that $(x) \sqsubseteq IN$

(since X is multiplication module). Therefore $x \in IN$, and hence $\sqsubseteq IN$, also it is clear that $IN \sqsubseteq N$. Thus from two above inclusions , we have $N = IN$, that is N is visible submodule.

$\Longrightarrow$) Let N be a visible submodule , to prove that $(N:X)$ is visible ideal. Let $x \in (N:_{T}X)$. Then $(x)X \sqsubseteq N$, implies $(x)X \sqsubseteq IN$ (since N is visible submodule). Then $(x)X \sqsubseteq I(N:X)X$. But X is cancellation module. Therefore $(x) \sqsubseteq I(N:X)$ and hence $(x) \in I(N:X)$.

Then $(N:X) \sqsubseteq I(N:X)$.

Conversely $I(N:X) \sqsubseteq (N:X)$. Therefore $(N:X) = I(N:X)$. This end the proof.

From proposition (1.1) , we obtain the following corollaries.

Corollary(1.2):

Let N be a proper submodule of a finitely generated faithful multiplication T-module X.Then N is visible if and only if $(N:X)$ is visible ideal of T.

Proof:

Since X is a finitely generated faithful multiplication module , then by ([10], proposition (3-1) , p.52) , we get X is cancellation and by proposition (1.1) we obtain the result.

We can introduce another proof for corollary (1.2) which not depend on proposition (1.1). But at first let us know the following:

A ring T (not necessary commutative) is called (Van Neumann) regular if $\forall t \in T , \exists a \in T$ such that $tat = t$, the purity property has been circulated to the modules by D. Field house [11] , a module X over T is called regular if each submodule W of X is pure in X , that is the inclusion $0 \longrightarrow W \longrightarrow X$ remains exact upon tensoring by any T-module .

Several definition about regular modules were discussed by Ware , Zelamanowitz , and Ramamurthi and Rangaswamy. Anderson and fuller in [9] named the submodule W a pure if $JW = W \cap JX$ for every ideal J of T.

Anther Proof of corollary (1.2):

Let N be a visible submodule of X . Then $N = IN$ for every nonzero ideal $I$of T

But N is pure submodule by ([1] , proposition(2.14)) , therefore $N \cap IX = IN$ for every ideal I of T. Then $N = N \cap IX$ and hence $\big((N \cap IX):X\big) = (N:X)$ which implies $(N:X) \cap (IX:X) = (N:X)$ [12]. Also from [6] , we get $(N:_{T}X)$ is pure ideal of T. Therefore $(N:_{T}X)(IX:_{T}X) = (N:X)$. Thus $(N:X)I = (N:X)$ by [10] , then $I(N:X) = (N:X)$. Thus $(N:X)$ is visible ideal of T.

Conversely :

Suppose that $(N:X)$ is visible ideal of T , then $(N:X) = I(N:X)$ for every nonzero ideal $I$of T. From ([1], proposition (2.14)) we obtain $(N:X)$ is pure ideal of T. Then $I \cap (N:X) = (N:X)$.

which implies $(IX:X) \cap (N:X) = (N:X)$.

And hence $\big((IX \cap N):X\big) = (N:X)$.

But X is multiplication module , then $\big((IX \cap N):X\big)X = (N:X)X$ which implies $IX \cap N = N$, then we get from [6] , N is pure , implies $IN = N$. Thus N is visible submodule

Corollary (1.3):

Let T be an integral domain and X be a faithful cyclic T-module Then N is a visible submodule of X if and only if $(N:X)$ is visible ideal of T.

Proof:

It is known that every cyclic module is multiplication , also every multiplication faithful module over integral domain is finitely generated , by ([10] , proposition (3-3) , p.54) and also by ([10], proposition (3-1) , p.52)) , we have X is cancellation module and by proposition (1.1) , we get the result.

A visible proper ideal J of T is defined as $J = AJ$ for each nonzero ideal A of T [1]. Now we have the following properties.

Proposition(1.4):

Let X be a finitely generated faithful multiplication T-module and I be a proper ideal of T. Then the following are hold:

(1) I is visible ideal of T $\Leftrightarrow$ IX is visible submodule of X.

(2) If N is visible submodule of , then $ann_{\mathrm{T}}(N) = ann_{\mathrm{T}}(N:X)$.

Proof:

(1). $\Longrightarrow$) Let I be a visible ideal of T. Then $JI = I$ for every nonzero ideal J of T and hence $JIX = IX$. Therefore IX is visible submodule.

$\Longleftarrow$) suppose that IX is a visible submodule of X then $JIX = IX$ for all proper ideal J of T (since X is finitely generated faithful multiplication module , then we obtain X is cancellation module by [10]). Therefore $JI = I$ and hence I is visible ideal of T.

(2). Let $x \in ann(N:X)$. Then $x(N:X) = 0$.

Which implies $N = x(N:X)N = 0$, therefore $x \in ann(N)$.

Now , let N be a visible submodule of X. Then $N = IN$ for every nonzero ideal I of T and by ([1] , proposition (2.14)) , we have N is pure , from this fact , we write $N = N \cap IX$ for every ideal I of T. But N is visible , therefore $IN = N \cap IX$. Taking $I = ann_{\mathrm{T}}(N)$ and hence $ann(N)N = N \cap ann(N)$.

$0 = N \cap ann(N)X$.

This leads us $(0:X) = ((N \cap ann(N)X:X)$

$= (N:X) \cap \ ann(N)X:X)$

$= (N:X) \cap (IX:X)$

$= (N:X) \cap I$

$= (N:X) \cap ann(N)$

$= (N:X)ann(N)$ by ([1] proposition (1.1) and proposition (2.14)).

Then $ann(X) = (N:X)ann(N)$.

But X is faithful which implies that $0 = (N:X)ann(N)$.

Therefore $ann(N) \sqsubseteq ann(N:X)$. Which complets the proof.

The following proposition introduce the necessary conditions for a visible submodule to be multiplication.

Proposition (1.5)

A visible submodule B of a finitely generated faithful multiplication T-module X is multiplication

Proof:

Let A be any submodule of X. Then $A = (A:X)X$ (since X is multiplication module) , we have B is visible submodule of X , then we get $B = IB$ for every a nonzero ideal I of T.

Hence $B \cap A \sqsubseteq B = IB = (A:B)B$ $(choose\ I = (A:B)$.

Therefore $B \cap A \sqsubseteq (A:B)B$ …(1).

Now , it is clear that $(A:B)B \sqsubseteq X$, then $(A:X)(A:B)B \sqsubseteq (A:X)X = A$.

Which implies that $((A:X)(A:B)B) \cap B \sqsubseteq A \cap B$. Hence $(A:B)(A:X)B \sqsubseteq A \cap B$. Now because B is visible submodule of X , then for every nonzero ideal I of T , we have $B = IB$ taking $I = (A:X)$, then

$(A:X)B = B$ and in the last we get $(A:B)B \sqsubseteq A \cap B$…(2).

From(1) and (2) then we obtain $A \cap B = (A:B)B$, that is B is multiplication submodule of X.

The next theorem provide equivalent statements for the visible submodules under certain conditions.

Theorem(1.6):

Suppose X is faithful finitely generated multiplication module over T ,D is proper submodule of X. So all will be equivalent:

1) D is visible submodule of X.
2) D is multiplication and is idempotent in X.
3) D is multiplication and $K = (D: X)K$ for each submodule K of D.
4) D is multiplication and $(K: D)D = (K: X)D$ for each submodule K of X.
5) $\mathrm{Td} = (D: X)d$ for each $d \in D$.
6) $\mathrm{T} = (D: X) + ann(d)$ for each $d \in D$.

Proof:

(1) $\Longrightarrow$ (2) From proposition (1.5) and ([1] ,proposition (2.18)) .

(2) $\Longrightarrow$ (3) Let D be a proper submodule of X then by (1) , D is multiplication submodule. For each submodule K of D , D is multiplication then $K = (K: D)D$.

Also X is multiplication so we will get $K = (K: D)(D: X)X$. But D is visible submodule , then by corollary (1.2) $(D: X)$ is visible ideal. Therefore $(K: D)(D: X) = (D: X)$ and hence $K = (D: X)X$.

This leads to $(K: X)K = (K: X)(D: X)X = (D: X)(K: X)X = (D: X)K$ (since X is multiplication , $K \leq X$).

K is visible because N is visible.

By ([1] , proposition (2.18)) , we have $K = (D: X)K$.

Permission (2) $\Longrightarrow$ (3) check.

(3) $\Longrightarrow$ (4) From (3) , we obtain directily D is multiplication. Also , we have D is visible submodule , then $D = ID$ for every nonzero ideal I of T (Taking $I = (K: X)$.

Therefore $D = (K: X)D$, also we can chose I another ideal of T , that is we can write $= (K: D)$, then $(K: D)D = D = (K: X)D$.

Therefore $(K: D)D = (K: X)D$.

(4) $\Longrightarrow$ (5) Since D is multiplication , then for every $d \in D$, we have $\mathrm{Td} = (\mathrm{D}: X)d$.

(5) $\Longrightarrow$ (6) by (5) , we have , for each $d \in D$ $\exists x \in (\mathrm{D}: X)$ $\ni d = xd$. Therefore $D = (x)D$ and hence $(x) = \mathrm{T}$ (since D is cancellation module as a result we get it from the fact that X is faithful FG and multiplication module).

Hence $(\mathrm{D}: X) = \mathrm{T}$ which implies $\mathrm{T} + \mathrm{ann}_{\mathrm{T}}(d) = (\mathrm{D}: X) + \mathrm{ann}_{\mathrm{T}}(d)$ and hence $\mathrm{T} = (\mathrm{D}: X) + \mathrm{ann}_{\mathrm{T}}(d)$. Therefore (6) holds.

(6) $\Longrightarrow$ (1) by (6) , we get T $D = (D: X)D + \mathrm{ann}_{\mathrm{T}}(d)D$ for each $d \in D$.

Therefore $D = (D: X)D$.

X is multiplication module , then $D = IX$ for some ideal I of T. Which implies $D = (IX: X)D$ (since X is cancellation). Therefore $D = ID$ and hence D is visible submodule of X.

Let us review the following properties.

Proposition (1.7):

Assume X is finitely generated faithful multiplication T-module and K is visible submodule of , then $\bigcap_{k\in I} J_k\, K = (\bigcap_{k\in I} J_k)K$, for every a nonempty collection J_k $(k \in I)$ of visible ideal of T.

Proof:

K is visible submodule of X , then by corollary (1.2)), we have $(K: X)$ is visible ideal of T. Suppose that J_k $(k \in I)$ is any collection of visible ideals of T. Now , $(\bigcap_{k\in I} J_k)K = K = (K: X)K$ by ([1] , proposition (2.18)) , which is equal

$(K: X)(\bigcap_{k\in I} J_k)K = (\bigcap_{k\in I} J_K)(K: X)K = (\bigcap_{k\in I} J_k)(K: X)AX$ for some ideal A of T. (since X is multiplication module)

we want to show that $(\bigcap_{k\in I} J_k\, K: X) = \bigcap_{k\in I} J_k\, (K \dot{:}_{\mathrm{T}} X)$ obviously , $\bigcap_{k\in I} J_k\, (K \dot{:}_{\mathrm{T}} X) \sqsubseteq (\bigcap_{k\in I} J_k\, K: X)$. Conversely let ,

$y \in (\bigcap_{k\in I} J_k\, K{:}X)$. Then $yX \sqsubseteq \bigcap_{k\in I} J_k\, K = \bigcap_{k\in I} J_k\, (K{:}X)$, but we have X is finitely generated multiplication module, then X is cancellation by [10]. Therefore $y \in \bigcap_{k\in I} J_k\, (K{:}X)$.

Now, $(\bigcap_{k\in I} J_k)(K{:}X)AX = (\bigcap_{k\in I} J_k\, K{:}X)AX$

$$= A(\bigcap_{k\in I} J_k\, K{:}X)X =$$

$A(\bigcap_{k\in I} J_k\, K)$

But J_k is visible ideal for all $k \in I$, then by ([1] , corollary(2.9)) we get $\bigcap_{k\in I} J_k$ is visible ideal, also by proposition (1.4) we obtain that $\bigcap_{k\in I} J_k\, K$ is visible, that is $A(\bigcap_{k\in I} I_k\, K) = \bigcap_{k\in I} I_k\, K$ $then$ $(\bigcap_{k\in I} J_k\, K) = \bigcap_{k\in I} J_k\, K$ and hence $(\bigcap_{k\in I} J_k)K = \bigcap_{k\in I} J_k\, K$.

Proposition(1.8):

Let X be a multiplication cancellation module over T , and K be a visible submodule of X. Then for each nonzero proper ideal E of T , result from this $E(K \dot{:}_{T} X) = (EK \dot{:}_{T} X)$.

Proof:

K is visible submodule , then $K = EK$ for each nonzero proper ideal E of T , implies $(K \dot{:}_{T} X) = (EK \dot{:}_{T} X)$, also by propositon (1.1), we get $(K \dot{:}_{T} X)$ is visible ideal of T.

Therefore $E(K \dot{:}_{T} X) = (K \dot{:}_{T} X)$ and hence $E(K \dot{:}_{T} X) = (EK \dot{:}_{T} X)$.

After giving above we can demonstrate proof of proposition (1.1) depending on proposition (1.8).

Proof:

$\Longrightarrow$) N is visible submodule of X , then for each a nonzero ideal I of T , we write $N = IN$, therefore $(N{:}X) = (IN{:}X)$ and by proposition (1.8) , we obtain $(N{:}X) = I(N{:}X)$. Thus we get the result.

$\Longleftarrow$) if $(N{:}X)$ visible then for each nonzero ideal I of T .

We have $(N{:}X) = I(N{:}X)$

And by proposition (1.8) , we obtain $(N{:}X) = (IN{:}X)$.

Therefore $(N{:}X)X = (IN{:}X)X$ and hence $N = IN$. Thus N is visible submodule.

Proposition (1.9):

Let X be a finitely generated faithful multiplication T-mduel and K be a visible submodule of X. Then $(K{:}X)$ is the intersection of all visible ideals I of T.

Proof:

Let A be a collection of visible ideals I of T. K is visible submodule of X , then $K = JK$ for every ideal $0 \neq J$ of T , and hence $K = IK$ where $I \in A$, therefore $\bigcap_{I\in A} I$ is visible ideal by ([1] , corollary (2.9)) , this lead us $K = \bigcap_{I\in A} I\, K = (\bigcap_{I\in A} I)K$ by proposition (1.7).

It follows that $(K{:}X) = \left((\bigcap_{I\in A} I)K{:}X\right) = (\bigcap_{I\in A} I)(K{:}X)$, but K is visible , then by proposition (1.1) , $(K{:}X)$ is visible and hence by ([1] , proposition (2.14)) , $(K{:}X)$ is pure , therefore

$(K{:}X) = (\bigcap_{I\in A} I) \cap (K{:}X)$ and hence $(K{:}X) \sqsubseteq \bigcap_{I\in A} I$, but K is visible , and hence an idempotent. Therefore $K = (K{:}X)K$.

It flows that $(K{:}X) \in A$. So $(K{:}X) = \bigcap_{I\in A} I$, therefore is the smallest element of A. This ends the proof.

Let's take the next result that shows that each submodule of fully cancellation module be a visible under condition that T is regular ring.

Proposition (1.10):

A proper submodule K of fully cancellation module X over a regular ring T will be visible.

Proof:

T is regular ring , then for every ideal G of T is pure , this leads us to $G^2 = G$ (since every pure ideal is idempotent). Let K be a proper submodule of X then $G^2K = GK$, which implies that $GGK = GK$ note GK , K are two distinct submodules of X and X is fully cancellation , this gives $GK = K$. Therefore K is visible.

Here , we will demonstrate the following results to reach to our important proposition.

Proposition(1.11):

Let T be a PIR and let X be a divisible T -module. Then each proper pure submodule of X is visible.

Proof:

Let I be a nonzero ideal of T and N be a proper pure submodule of X. Since T is PIR , then $\mathrm{I} = (\mathrm{r})$ for some $r \in \mathrm{T}$, $\mathrm{r} \neq 0$.

We must prove that $N = IN$. It is clearly that $IN \subseteq N$, to prove the another inclusion (that is $N \subseteq IN$) let $n \in N$. Then $n \in N \cap X$. But X is divisible , then $X = rX$ for all $r \in \mathrm{T}$, $\mathrm{r} \neq 0$. Therefore $n \in N \cap rX$ which implies that $n \in rN$ (since N is pure submodule).

Therefore $n \subseteq rN$ and hence $n \subseteq IN$, next we obtain $N \subseteq IN$.

<u>Proposition(1.12):</u>

Let X ba a divisible module over a PIR and H be a proper submodule of X. Then the following hold:

(1). If M/H is flat , then H is visible submodule and the converse hold when X is a flat T-module.

(2). If H is a visible submodule of a flat module , then H is flat.

<u>Proof:</u>

(1). From proposition (1.11) and ([13] proposition (2.3),p.20). we get the result of number (1).

(2). From proposition (1.11) and([13], proposition (3.3), p.22). we get the submodule H is flat.

<u>Proposition(1. 13):</u>

Let X be a divisible module over a PIR and H be a proper submodule of X. If X is a multiplication faithful T-module , then H is flat.

Proof:

Since X is multiplication faithful T-module Then $X/H$is also multiplication faithful T-module Then by [14] we obtain that X/H is flat T-module and by the first side of (1) of proposition (1.12) , we get H is visible submodule , and by (2) of proposition (1.12) , we obtain H is flat (since X is multiplication faithful T-module) , then X is flat by [14].

<u>2.Trace of visible submodules</u>

The trace of visible submodule of X over T has been studied here and symbolizes it by $Tr(W)$ and the set $\{\sigma(w) : \sigma \in Hom(W, \mathrm{T}) , \mathrm{w} \in \mathrm{W}\}$, is a set of generator for $Tr(W)$.

Two important descriptions for the trace of visible submodule of multiplication module have been given , also has been proven when the visible submodules of multiplication modules are torsionless. Where a module Xover T is named torsionless , if X can be embedded in direct product of copies of T [15] , add to that many properties of $Tr(W)$ have been presented.

<u>Proposition (2.1):</u>

If N is a visible submodule of a finitely generated faithful multiplication T-module , then $(N : X) = Tr(N) = \sum_{a \in N} ann(ann(a))$.

Proof:

Suppose that N is visible submodule of , then for each nonzero ideal I of T , we have $N = IN$. Taking $I = (N \dot{:}_{\mathrm{T}} X)$, then $N = (N : X)X$. Therefore $\forall \theta \in Hom(N, \mathrm{T})$, $\theta(N) = \theta((N \dot{:}_{\mathrm{T}} X)N) = (N \dot{:}_{\mathrm{T}} X)\theta(N)$.

Therefore $\sum_{\theta} \theta(N) = (N : X) \sum_{\theta} \theta(N)$ and hence $Tr(N) = (N : X)Tr(N)$, but X is finitely generated faithful multiplication and a submodule N of X is pure by ([1] , proposition (2,14)) , then $(N : X)$ is pure ideal of T , that is

$$Tr(N) = (N : X)Tr(N)$$
$$= (N : X) \cap Tr(N)$$
$$= (N : X) \cap (Tr(N)X : X)$$
$$= ((N \cap Tr(N)X : X)$$
$$= (Tr(N)N : X)$$
$$= (N : X) \text{ (since } N \text{ is visible).}$$

Therefore $Tr(N) = (N:X)$.

Suppose now that $a \in N$ and $\theta \in Hom(N, \mathrm{T})$.

Clearly $ann(a) \sqsubseteq ann(\theta(a))$. Therefore $ann(a)\theta(a) = 0$ and hence $\theta(a) \in ann(ann(a))$.

Which implies $\mathrm{T}\theta(\mathrm{a}) \sqsubseteq \mathrm{ann}(\mathrm{ann}(\mathrm{a})$ and $\theta(N) = \sum_{a\in N} \mathrm{T}\theta(\mathrm{a}) \sum_{a\in N} ann(ann(a))$.

Therefore $Tr(N) \sqsubseteq \mathrm{ann}(\mathrm{ann}(\mathrm{a}))$ and hence $(N:X) \sqsubseteq \mathrm{ann}(\mathrm{ann}(\mathrm{a}))$.

Another side , let $a \in N$.

Then by (theorem (1,6), (6)) we get $\mathrm{T} = (\mathrm{N}:X) + ann(a)$.

Therefore $\mathrm{T}\,\mathrm{ann}(\mathrm{ann}(\mathrm{a})) = (\mathrm{N}:X)ann(ann(a)) + ann(ann(a))ann(a)$ And hence $ann(ann(a)) = (N:X)ann(ann(a)$.

which implies that $ann(ann(a) = Tr(N)ann(ann(a) \sqsubseteq \mathrm{Tr(N)} = (\mathrm{N}:X)$. Thus $ann(ann(a) \sqsubseteq Tr(N)$. Then we get $Tr(N) = \sum_{a\in N} ann(ann(a))$. This lead us to write $Tr(N) = (\mathrm{N}:X) = \sum_{a\in N} ann(ann(a))$.

Next we review the most important application for proposition (2.1).

Corollary (2.2):

If X is faithful finitely generated multiplication generated module on T and W is visible submodule of , then $Tr(D) = (D \dot{\cdot}_{\mathrm{T}} X\ ) = ann_{\mathrm{T}}(ann_{\mathrm{T}}(D))$.

Proof:

By proposition (2.1) , we achieved the first equality and represented by $Tr(D) = (D \dot{\cdot}_{\mathrm{T}} X\ )$. The rest is to prove that $(D \dot{\cdot}_{\mathrm{T}} X\ ) = ann_{\mathrm{T}}(ann_{\mathrm{T}}(D))$.

For ease we will write $Ann(D) = ann_{\mathrm{T}}(ann_{\mathrm{T}}(D))$. We want to prove that $Ann(D = (D \dot{\cdot}_{\mathrm{T}} X\ )$. As D is visible submodule of , then from theorem (2.1) , we have $\mathrm{T} = (D \dot{\cdot}_{\mathrm{T}} X\ ) + ann(N)$.

Therefore $Ann(D) = (D \dot{\cdot}_{\mathrm{T}} X\ )Ann(D)$ and hence $Ann(D) \subseteq (D \dot{\cdot}_{\mathrm{T}} X\ )$. Now , let $y \in (D \dot{\cdot}_{\mathrm{T}} X\ )$. Then $yX \subseteq D$ which implies $y\, ann(N)X = 0$. Therefore $y\, ann(D) \subseteq ann(X) = 0$ (since X is faithful).

Thus $(D \dot{\cdot}_{\mathrm{T}} X\ ) \subseteq Ann(D)$ so that $Ann(D) = (D \dot{\cdot}_{\mathrm{T}} X\ )$This gives the end of the proof.

Corollary (2.3):

Suppose X is finitely generated faithful multiplication T-module and N is visible submodule of , then

(1) $N = Tr(N)N$.

(2) $ann(N) = ann(Tr(N))$.

Proof:

(1) According to proposition (2.1) , we get $Tr(N) = (N:X)$, then $Tr(N)N = (N:X)N$. But every visible module is an idempotent by ([1] , proposition (2.18)). Therefore $Tr(N)N = N$.

(2) Suppose $r \in ann_{\mathrm{T}}(N)$, then $rN = 0$ implies $\theta_i(rN) = 0$ and hence $\sum_{i=1}^{n} \theta_i(rN) = 0$ this gives $Tr(N) = 0$, so that $r \in ann_{\mathrm{T}}(Tr(N)$ and $ann_{\mathrm{T}}(N) \sqsubseteq ann_{\mathrm{T}}(Tr(N)$.

Let $r \in ann(Tr(N))$.

Then by proposition (2.1) we obtain that $r \in ann_{\mathrm{T}}(N:X)$, and by proposition (1.4), we obtain $r \in ann_{\mathrm{T}}(N)$. This gives the other direction of containment. Thus (2) holds.

Low and smith in [16] Demonstrated that for a multiplication faithful module X if $f \epsilon Hom(X,T)$,$\cap\, ker f = 0$ then X is torsionless.

Corollary (2.4):

Let X be a finitely generated faithful multiplication module over T and D be a visible submodule of X. Then D is torsionless .

Proof:

Suppose that $H = \cap_{\sigma \in Hom(D,\mathrm{T})} Ker\sigma$. Then from proposition (1,5) , D is multiplication , therefore $H = (H:D)D$, follow from this

$0 = \sigma(H) = (H:D)\sigma(D)$ for all $\sigma \in Hom(D, \mathrm{T})$.

Implies that $0 = (H\colon D)Tr(D)$.

Therefore by proposition (2.1) and corollary (2.3) obtain that $(H\colon D) \sqsubseteq ann_{\mathrm{T}}(Tr(D) = ann_{\mathrm{T}}([D\colon X]) = ann_{\mathrm{T}}(D)$.

Finally $H = (H\colon D)D = 0$. That is the answer.

The coming result of the item offers important properties for the trace of visible module.

Corollary (2.5):

Let X be a faithful finitely generated multiplication module over T and D is visible submodule of X. Let $D = H \oplus L$ for two submodules H and L of X. Then

1) $Tr(D) = Tr(H) \oplus Tr(L)$.

2) $D = Tr(H)D \oplus Tr(L)D$.

Proof:

(1).Since D is visible submodule of X , then H , L are also visible submodules by ([1], proposition (2.7)) and by proposition (2.1) , we obtain

$Tr(D) = (D\colon X)$, $Tr(H) = (H\colon X)$, $Tr(L) = (L\colon X)$.
Therefore by [17] and , ([13], proposition (4)) we obtain
$Tr(D) = (H + L\colon X) = (H\colon X) + (L\colon X) = Tr(H) + Tr(L)$.

Since X is faithful , then

$0 = (0 :_{\mathrm{T}} X) = (H \cap L\colon X) = (H\colon X) \cap (L\colon X) = Tr(H) \cap Tr(L)$.

Hence $Tr(D) = Tr(H) \oplus Tr(L)$.

(2). From corollary (2.3), we have

$D = Tr(D)D$ and by (1) , we get $D = Tr(H)D + Tr(L)D$, also by ([1] , proposition (2.13)) , we have

$Tr(H)D \cap Tr(L)D = \big(Tr(H) \cap Tr(L)\big)D = 0$.

Thus $D = Tr(H)D \oplus Tr(L)D$.

<u>Proposition (2.6):</u>

Let X be a finitely generated faithful multiplication over T , N is visible submodule of X. Then $Tr(N)$ is pure ideal of T.

Proof:

We own N is visible submodule , then for each nonzero ideal A of T , $N = AN$. Thus we take $\theta_i(N) = \theta_i(AN)$.

Therefore $\sum_{i=1}^{n} \theta_i (N) = \sum_{i=1}^{n} \theta_i (AN) = \sum_{i=1}^{n} A\theta_i (N) = A \sum_{i=1}^{n} \theta_i (N)$ and hence $Tr(N) = A\, Tr(N)$.

Now to prove that $Tr(N)$ is pure ideal.

Case (1): $A \cap Tr(N) \sqsubseteq Tr(N) = A\, Tr(N)$ as well as the second case verified and this means $A\, Tr(N) \sqsubseteq A \cap Tr(N)$.

Therefore $A\, Tr(N) = A \cap Tr(N)$. That invites us to say that $Tr(N)$ is pure ideal.

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On μ*-extending modules

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Abstract: Let R be an associative ring with identity and let M be a left R- module. As a generalization of essential submodules Zhou defined an Ғ- essential submodules provided it has a nonzero intersection with any nonzero submodule in Ғ where Ғ is a collection of R- modules such that if $M \in$ Ғ , then $M' \in$ Ғ for any module M' isomorphic to M. In this article we study μ*- essential submodules as a dual of μ-small submodules provided it has a nonzero intersection with any nonzero singular submodule of M. Also we define and investigate μ*-extending modules with some examples and basic properties.

Keywords. μ*-essential, μ*-closed submodules, μ*-extending modules.

1. Introduction

Let R be an associative ring with unity and let M be unitary left R- module. A submodule A of M is said to be essential in M, (denoted by $A \leq_e M$), if for any submodule B of M , $A \cap B = 0$ implies $B = 0$ [1], and a submodule A of M is said to be closed in M if A has no proper essential extension in M; that is if $A \leq_e B \leq M$, then $A = B$ [1]. An R-module M is called extending (or CS- module) , if every submodule of M is essential in a direct summand of M. It is well known that an R- module M is extending if and only if every closed submodule of M is a direct summand [2]. A submodule A of M is called μ- small submodule of M (denoted by $A \ll_\mu M$) if whenever $M = A + X$, $\frac{M}{X}$ is cosingular , then $M = X$, see [3]

Following [4], Zhou defined an Ғ- essential submodules provided it has a nonzero intersection with any nonzero submodule in Ғ where Ғ is a collection of R- modules such that if $M \in$ Ғ , then $M' \in$ Ғ for any module M' isomorphic to M. In this paper we introduce μ*- essential submodules as a dual of μ-small submodules provided it has a nonzero intersection with any nonzero singular submodule of M.

An R- module M is called μ*- extending module if every submodule of M is μ*- essential in a direct summand.

In section two , we define and study μ*-essential submodules , μ*- closed submodules and μ*- uniform modules.

In section three , we introduce μ*- extending modules with some examples and basic properties , we give sufficient conditions for a submodules of μ*- extending modules to be μ*- extending module.

In section four , we give various characterizations of μ*- extending modules and study the direct sum of μ*- extending modules.

2. μ*-essential and μ*- closed submodules.

In this section, we introduce μ*- essential submodules and μ*- uniform modules as a generalization of essential submodules and uniform modules respectively which are duals of μ- small submodules and μ- hollow modules. Also , we define a μ*- closed submodules which is stronger than closed submodules. We study the basic properties of them that are relevant to our work.

Definition (2.1): Let A be a submodule of an R- module M, M is said to be **μ*-essential extension** to A or A is a **μ*-essential in M** if for any nonzero singular submodule B of M , we have $A \cap B \neq 0$. It will be denoted by $A \leq_{\mu^* e} M$.

Remarks and Examples (2.2).

(1) It is clear that μ*- essential submodules are generalizations of essential submodules. There is a μ*- essential submodule of an R- module M which is not essential in M. For example: Consider Z_6 as Z_6- module . Since Z_6 is nonsingular Z_6- module , then $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are μ*- essential in Z_6 which are not essential in Z_6.

(2) Every nonzero submodule of Q as Z- module is μ*- essential in Q.

(3) Every nonzero cyclic submodule of Z as Z- module is μ*- essential in Z.

(4) Consider Z_6 as Z- module , $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are not μ*- essential in Z_6.

In the following propositions we consider conditions under which μ*-essential submodules versus essential submodules.

Proposition(2.3): Let M be a singular R- module and let A be a submodule of M , then $A\leq_{\mu^*e} M$ if and only if $A\leq_e M$.

Proof: It is clear.

Let R be a commutative integral domain and M be an R-module. Recall that $T(M) = \{m\in M: rm = 0$, for some nonzero $r\in R\}$ is called the torsion submodule of M. If $T(M) = M$ (if $T(M) = 0$), then M is called **torsion (torsion free) module**, see [5].

Proposition (2.4): Let M be a torsion module over a commutative integral domain R and A be a submodule of M. Then $A\leq_{\mu^*e} M$ if and only if $A\leq_e M$.

Proof: It is clear by [5, P. 31] and Prop. (2.3).

Let M be an R-module . Recall that M is called a **prime** R- module if $ann(x) = ann(y)$, for every nonzero elements x and y in M , see [6].

Proposition (2.5): Let M be a prime R- module with $Z(M)\neq 0$ and A be a submodule of M. Then $A\leq_{\mu^*e} M$ if and only if $A\leq_e M$.

Proof: Assume that $A\leq_{\mu^*e} M$. To show that M is singular . Let $0\neq x\in Z(M)$, then $ann(x)\leq_e R$ and let $0\neq y\in M$. Since M is prime module , then $ann(x) = ann(y)$ and hence $y\in Z(M)$. Thus $Z(M) = M$ and hence $A\leq_e M$, by Prop. (2.3). The proof of the converse is clear. □

Next, we give characterizations of μ*- essential submodules.

Proposition (2.6): Let M be an R- module and let A be a submodule of M , then $A\leq_{\mu^*e} M$ if and only if for any nonzero cyclic singular submodule K of M , $A\cap K\neq 0$.

Proof: Let K be a nonzero cyclic singular submodule of M and let $0\neq x\in K$. By our assumption $0\neq \langle x\rangle\cap A\leq A\cap K$. Hence $A\cap K\neq 0$. The proof of the converse is clear. □

Proposition (2.7): Let M be an R- module and let A be a submodule of M , then $A\leq_{\mu^*e} M$ if and only if for any nonzero element x in M with Rx singular has a nonzero multiple in A.

Proof: Let $0\neq x\in M$ with Rx singular submodule of M. By Prop. (2.6) $Rx\cap A\neq 0$. Hence there is $r\in R$ such that $0\neq rx\in A$. The proof of the converse is clear. □

Proposition (2.8): Let M be any R- module. Then the following are hold.

(1) Let submodules $A\leq B\leq M$. Then $A\leq_{\mu^*e} M$ if and only if $A\leq_{\mu^*e} B$ and $B\leq_{\mu^*e} M$.

(2) Let $A_1\leq_{\mu^*e} B_1\leq M$ and $A_2\leq_{\mu^*e} B_2\leq M$, then $A_1\cap A_2 \leq_{\mu^*e} B_1\cap B_2$.

(3) If $f: M_1\rightarrow M_2$ is an R- homomorphism and $A\leq_{\mu^*e} M_2$, then $f^{-1}(A)\leq_{\mu^*e} M_1$.

(4) Let $\{A_\alpha\}\ \alpha\in\Lambda$ be an independent family of submodules of M and $A_\alpha\leq_{\mu^*e} B_\alpha$, $\forall\alpha\in\Lambda$, then $\bigoplus_{\alpha\in\Lambda} A_\alpha\leq_{\mu^*e} \bigoplus_{\alpha\in\Lambda} B_\alpha$.

Proof. (1) Suppose that $A\leq_{\mu^*e} M$ and let L be a nonzero singular submodule of B. Since $A\leq_{\mu^*e} M$, then $A\cap L\neq 0$. Hence $A\leq_{\mu^*e} B$. Now let K be a nonzero singular submodule of M , then $0\neq A\cap K\leq B\cap K$. Thus $B\leq_{\mu^*e} M$.

Conversely , assume that $A\leq_{\mu^*e} B\leq_{\mu^*e} M$ and let L be a nonzero singular submodule of M , then $B\cap L$ is a nonzero singular submodule of B. But $A\leq_{\mu^*e} B$, therefore $A\cap B\cap L = A\cap L\neq 0$. Thus we get the result.

(2) Assume that $A_1\leq_{\mu^*e} B_1\leq M$ and $A_2\leq_{\mu^*e} B_2\leq M$ and let L be a nonzero singular submodule of $B_1\cap B_2\leq B_1$. Since $A_1\leq_{\mu^*e} B_1$, then $A_1\cap L\neq 0$ and hence it is a nonzero singular submodule of B_2. But $A_2\leq_{\mu^*e} B_2$, therefore $A_1\cap A_2\cap L\neq 0$. Thus $A_1\cap A_2\leq_{\mu^*e} B_1\cap B_2$.

(3) Let $f: M_1\rightarrow M_2$ be an R- homomorphism and let $A\leq_{\mu^*e} M_2$. To show that $f^{-1}(A)\leq_{\mu^*e} M_1$, let $0\neq x\in M_1$ with Rx is singular submodule of M_1, then $f(Rx)$ is a singular submodule of M_2. Consider the following two cases.

(a) if $x\in f^{-1}(A)$, we are done.

(b) if $x\notin f^{-1}(A)$, $0\neq f(x)\in M_2$. Since $A\leq_{\mu^*e} M_2$, then there is $r\in R$ such that $0\neq rf(x)\in A$, hence $0\neq rx\in f^{-1}(A)$. Thus $f^{-1}(A)\leq_{\mu^*e} M_1$.

(4) We use the induction on the number of elements of Λ. Suppose that the family has only two elements. i.e. , $\{A_1, A_2\}$ is independent family in M, $A_1\leq_{\mu^*e} B_1$ and $A_2\leq_{\mu^*e} B_2$. Let $\pi_1: B_1\oplus B_2\rightarrow B_1$ and $\pi_2: B_1\oplus B_2\rightarrow B_2$ be the projection maps. Since $A_1\leq_{\mu^*e} B_1$ and $A_2\leq_{\mu^*e} B_2$, then $\pi_1^{-1}(A_1) = A_1\oplus B_2\leq_{\mu^*e} B_1\oplus B_2$ and $\pi_2^{-1}(A_2) = B_1\oplus A_2\leq_{\mu^*e} B_1\oplus B_2$, by(3) and hence $A_1\oplus A_2 = (A_1\oplus B_2)\cap(B_1\oplus A_2)\leq_{\mu^*e} B_1\oplus B_2$, by (2).

Now, assume that the result is true for the case when the index set with $n-1$ elements. Now let $\{A_1, A_2, \ldots, A_n\}$ be an independent family and assume that $A_i \leq_{\mu^*e} B_i$, $\forall$ i = 1, 2,….,n. By the previous case we have $\bigoplus_{i=1}^{n-1} A_i \leq_{\mu^*e} \bigoplus_{i=1}^{n-1} B_i$ and $A_n \leq_{\mu^*e} B_n$, hence we get $\bigoplus_{i=1}^{n} A_i \leq_{\mu^*e} \bigoplus_{i=1}^{n} B_i$. Finally, let $\{A_\alpha\}$ $\alpha \in \Lambda$ be an independent family of submodules of M and $A_\alpha \leq_{\mu^*e} B_\alpha$, $\forall \alpha \in \Lambda$. Let N be a nonzero singular submodule of $\bigoplus_{\alpha \in \Lambda} B_\alpha$ and let x be a nonzero element in N. So x = $b_1+b_2+\ldots+b_n$, where $b_i \in B_{\alpha_i}$, $\forall$ i = 1,2,…,n. Hence $N \cap (A_{\alpha 1}+ A_{\alpha 2}+\ldots+ A_{\alpha n}) \neq 0$ which implies that $N \cap \bigoplus_{\alpha \in \Lambda} A_\alpha \neq 0$. Thus $\bigoplus_{\alpha \in \Lambda} A_\alpha \leq_{\mu^*e} \bigoplus_{\alpha \in \Lambda} B_\alpha$.

□

Notes. (1) Note that $\{B_\alpha\}_{\alpha \in \Lambda}$ in proposition (2.8-4) need not be an independent family. Example: Let M be the Z-module $Z \oplus Z_2$ and let $A_1 = 0 \oplus Z_2$, $B_1 = Z \oplus Z_2$, $A_2 = B_2 = Z \oplus \bar{0}$. One can easily show that $A_1 \leq_{\mu^*e} B_1$ and $A_2 \leq_{\mu^*e} B_2$ and $A_1 \cap A_2 = \{0\}$ but $B_1 \cap B_2 = Z \oplus \bar{0}$. Hence $\{B_1,B_2\}$ is not independent family.

(2) Let A_1, A_2, B_1 and B_2 be submodules of an R- module M. If $A_1 \leq_{\mu^*e} B_1$ and $A_2 \leq_{\mu^*e} B_2$, then it is not necessary that $(A_1+A_2) \leq_{\mu^*e} (B_1+B_2)$ as the following example shows:

Consider the Z- module $Z \oplus Z_2$. Let $A_1 = A_2 = Z(\bar{2}, \bar{0})$ and $B_1 = Z(\bar{1},\bar{0})$, $B_2 = Z(\bar{1},\bar{1})$. One can easily show that $A_1 \leq_{\mu^*e} B_1$ and $A_2 \leq_{\mu^*e} B_2$. But (A_1+B_1) is not μ*-essential in (B_1+B_2) , where there exists a nonzero singular submodule $K = \{\bar{0}\} \oplus Z_2$ of (B_1+B_2) such that $(A_1+A_2) \cap K = \{(\bar{0},\bar{0})\}$.

Recall that a submodule A of an R- module M is called a **closed submodule** of M if A has no proper essential extension. See [1].

Now, we define the μ*- closed submodules and introduce the basic properties of these submodules.

Definition (2.9): Let A be a submodule of an R- module M, we say that A is **μ*-closed in *M*** (briefly $A \leq_{\mu^*c} M$) if A has no proper μ*- essential extension in M.

The following proposition ensure the existences of μ*-closed submodules.

Proposition (2.10): Let M be an R- module . Then every submodule is μ*- essential in μ*- closed submodule of M.

Proof: Let A be a submodule of M. Consider the collection $\Gamma = \{K: K \leq M: A \leq_{\mu^*e} K\}$. It is clear that Γ is nonemplty set . Let $\{C_\alpha\}$ $\alpha \in \Lambda$ be a chain in Γ. To show that $A \leq_{\mu^*e} \bigcup_{\alpha \in \Lambda} C_\alpha$, let $0 \neq x \in \bigcup_{\alpha \in \Lambda} C_\alpha$ with Rx is singular submodule of $\bigcup_{\alpha \in \Lambda} C_\alpha$, then there is $\alpha_\circ \in \Lambda$ such that $0 \neq x \in C_{\alpha_\circ}$ .But $A \leq_{\mu^*e} C_\alpha$, $\forall \alpha \in \Lambda$, therefore there exists $r \in R$ such that $0 \neq rx \in A$, hence $A \leq_{\mu^*e} \bigcup_{\alpha \in \Lambda} C_\alpha$ which means that $\bigcup_{\alpha \in \Lambda} C_\alpha \in \Gamma$. By Zorn's lemma Γ has a maximal element say H. To show that H is μ*- closed in M , let B be a submodule of M such that $H \leq_{\mu^*e} B$, then $A \leq_{\mu^*e} H \leq_{\mu^*e} B$ and hence $A \leq_{\mu^*e} B$, by Prop. (2.8) . But H is maximal element in Γ. Thus H = B.

□

Remarks and Examples (2.11).

(1) Every μ*- closed submodule of an R- module M is closed in M. The converse is not true in general. For example , Consider Z_6 as Z_6- module $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are closed in Z_6 but not μ*- closed in Z_6.

(2) Consider Z_6 as Z- module , $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are μ*-closed submodules of Z_6.

(3) In Z_4 as Z- module , $\{\bar{0},\bar{2}\}$ is not μ*- closed in Z_4.

(4) Let M be a singular R- module. Then A is closed in M if and only if A is μ*- closed in M.

(5) Let M be a torsion module over a commutative integral domain R and A be a submodule of M. Then $A \leq_{\mu^*c} M$ if and only if $A \leq_c M$.

(6) Let M be a prime R- module with $Z(M) \neq 0$ and A be a submodule of M. Then $A \leq_{\mu^*c} M$ if and only if $A \leq_c M$.

(7) It is well known that every direct summand of an R-module M is closed in M. But in case μ*-closed there is no relationship with direct summands. For example , Z_6 as Z_6- module , the nontrivial direct summands of Z_6 are $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ which are not μ*- closed in Z_6.

(8) If a submodule A of an R- module M is μ*- closed and μ*- essential in M , then $A = M$.

(9) The intersection of μ*- closed submodules of M need not be μ*- closed in M. For example , consider $M = Z \oplus Z_2$ as Z- module , let $A = Z \oplus \bar{0}$, $B = Z(1,\bar{1})$. Since $0 \oplus Z_2$ is the only singular submodule of M and has zero intersection with A , then $A \leq_{\mu^*c} M$. Similarly $B \leq_{\mu^*c} M$, but $A \cap B = 2Z \oplus \bar{0}$ which is not μ*- closed in M.

Next, we give the basic properties of μ*-closed submodules.

Proposition (2.12): Let M be an R- module. If $A \le_{\mu^*c} M$, then $\frac{B}{A} \le_{\mu^*e} \frac{M}{A}$, whenever $B \le_{\mu^*e} M$ with $A \le B$.

Proof. Suppose that $A \le B \le_{\mu^*e} M$ and let $\frac{L}{A}$ be a singular submodule of $\frac{M}{A}$ such that $\frac{L}{A} \cap \frac{B}{A} = A$, then $L \cap B = A$. Since $B \le_{\mu^*e} M$, then $A \le_{\mu^*e} L$, by Prop. (2.8-2). But A is μ*- closed in M, therefore $A = L$. Thus $\frac{B}{A} \le_{\mu^*e} \frac{M}{A}$. □

Proposition (2.13): Let $f : M \to M'$ be an epimorphism and let A be a submodule of M such that $Kerf \le A$. If A is μ*- closed in M, then $f(A)$ is μ*- closed in M'.

Proof. Let K' be a submodule of M' such that $f(A) \le_{\mu^*e} K'$, then $f^{-1}(f(A)) \le_{\mu^*e} f^{-1}(K')$, by Prop. (2.8). One can easily show that $f^{-1}(f(A)) = A$, hence $A \le_{\mu^*e} f^{-1}(K')$. But A is μ*- closed in M, therefore $A = f^{-1}(K')$, and hence $f(A) = K'$. Thus $f(A)$ is μ*- closed in M'. □

One can easily prove the following corollaries.

Corollary (2.14): μ*- closed submodule is closed under isomorphism.

Corollary (2.15): Let A and B be submodules of an R- module M with $A \le B$. If B is μ*- closed in M, then $\frac{B}{A}$ is μ*- closed in $\frac{M}{A}$.

Proposition (2.16): Let M be an R- module and let A, B be submodules of M with $A \le B \le M$. If A is μ*- closed in M, then A is μ*- closed in B.

Proof: Suppose that $A \le_{\mu^*e} L \le B \le M$. But A is μ*- closed in M, therefore $A = L$. Thus A is μ*- closed in B. □

It is easy to prove the following corollary.

Corollary (2.17): Let A and B be submodules of an R- module M if $A \cap B$ is μ*- closed in M, then $A \cap B$ is μ*- closed in A and B.

We cannot prove the transitive property for μ*- closed submodules. However under certain condition we can prove this property as we see in the following result.

Recall that an R- module M is called **chained module** if for each submodules A and B of M either $A \le B$ or $B \le A$, see [7].

Proposition (2.18): Let M be chained R- module and let A and B be submodules of M such that $A \le B \le M$. If $A \le_{\mu^*c} B \le_{\mu^*c} M$, then $A \le_{\mu^*c} M$.

Proof. Let K be a submodule of M such that $A \le_{\mu^*e} K \le M$. By our assumption we have two cases: If $K \le B$. Since A is μ*- closed in B, then $A = K$, hence $A \le_{\mu^*c} M$. If $B \le K$, since $A \le_{\mu^*e} K$, so $B \le_{\mu^*e} K$, by Prop. (2.8). But $B \le_{\mu^*c} M$, therefore $B = K$, hence $A \le_{\mu^*e} B$. But $A \le_{\mu^*c} B$, therefore $A = B = K$. Thus A is μ*- closed in M. □

The following proposition shows that the direct sum of μ*-closed submodules is again μ*- closed.

Proposition (2.19): Let M_1, M_2 be two R- modules. If $A_1 \le_{\mu^*c} M_1$ and $A_2 \le_{\mu^*c} M_2$, then $A_1 \oplus A_2 \le_{\mu^*c} M_1 \oplus M_2$.

Proof: Assume that $A_1 \oplus A_2 \le_{\mu^*e} B_1 \oplus B_2$, $B_1 \le M_1$ and $B_2 \le M_2$, let $i_1: M_1 \to M_1 \oplus M_2$ and $i_2: M_2 \to M_1 \oplus M_2$ be the inclusion maps. Since $A_1 \oplus A_2 \le_{\mu^*e} B_1 \oplus B_2$, then $i_1^{-1}(A_1 \oplus A_2) \le_{\mu^*e} i_1^{-1}(B_1 \oplus B_2)$. Note that $i_1^{-1}(A_1 \oplus A_2) = \{x \in M_1: i_1(x) \in (A_1 \oplus A_2)\} = \{x \in M_1: (x,0) \in (A_1 \oplus A_2)\} = A_1 \le_{\mu^*e} i_1^{-1}(B_1 \oplus B_2) = B_1$. Similarly, $A_2 \le_{\mu^*e} B_2$. But $A_1 \le_{\mu^*c} M_1$ and $A_2 \le_{\mu^*c} M_2$, therefore $A_1 = B_1$ and $A_2 = B_2$. Thus $A_1 \oplus A_2 \le_{\mu^*c} M_1 \oplus M_2$. □

An R- module M is called **uniform** module if every nonzero submodule of M is essential in M, see [1].

Now, we introduce μ*- uniform modules as a generalization of uniform modules which is a dual of μ-hollow modules.

Definition (2.20): An R- module M is called **μ*- uniform** if every nonzero submodule of M is μ*- essential in M.

Remarks and Examples (2.21):

(1) Every nonsingular module is μ*- uniform. The converse is not true in general, for example, Z_4 as Z- module.
(2) Every torsion free module over a commutative integral domain is μ*- uniform.
(3) Clearly that every uniform module is μ*- uniform, hence Q as Z- module and Z- as Z- module are μ*- uniform modules.
(4) The converse of (3) is not true in general. For example, Z_6 as Z_6- module.
(5) Z_6 as Z- module is not μ*- module.
(6) Let M be a singular R- module. Then M is uniform if and only if M is μ*- uniform.

(7) Let M be a torsion module over a commutative integral domain R. Then M is uniform if and only if M is μ*- uniform.

(8) Let M be a prime R- module with $Z(M) \neq 0$. Then M is uniform if and only if M is μ*- uniform.

The following theorem gives a characterization of μ*- uniform modules. Compare with [3, theorem (3.7)].

Proposition (2.22): Let M be an R- module. Then M is μ*- uniform if and only if every nonzero singular submodule of M is essential in M.

Proof: ($\Rightarrow$) Assume that M is μ*- uniform and let A be a nonzero singular submodule of M. Assume that there exists a nonzero submodule B of M such that $A \cap B = 0$. Since M is μ*- uniform , then $B \leq_{\mu^*e} M$ and we have A is nonzero singular submodule of M , then $A \cap B \neq 0$, which is a contradiction.

($\Leftarrow$) To show that M is μ*- uniform , let A be a nonzero submodule of M and assume that A is not μ*- essential in M , that is there exists a nonzero singular submodule B of M such that $A \cap B = 0$. By our assumption $B \leq_e M$, then $A = 0$, which is a contradiction. □

Compare the following Prop. with [3, Prop. (3.8)]

Proposition (2.23): A nonzero monomorphic image of μ*- uniform is μ*- uniform.

Proof: Let $f : M \to M'$ be an R- monomorphism and assume that M is μ*- uniform , we have to show that M' is μ*- uniform , let A be a nonzero submodule of M', then $f(A) \neq 0$, if $f(A) = 0$, then $A \leq Ker f = 0$ which is a contradiction. Since M' is μ*- uniform , then $f(A) \leq_{\mu^*e} M'$ and hence $A \leq_{\mu^*e} M$. □

Corollary (2.24): A submodule of μ*- uniform is again μ*- uniform.

Note. A quotient of μ*- uniform need not be μ*- uniform. For example , Z as Z- module is μ*- uniform but $\frac{Z}{6Z} \cong Z_6$ which is not μ*- uniform.

The following proposition gives a condition under which a quotient of μ*- uniform is μ*- uniform.

Proposition (2.25): Let M be a μ*- uniform and let A be a μ*- closed submodule of M , the n $\frac{M}{A}$ is μ*- uniform.

Proof: Let $\frac{L}{A}$ be a nonzero submodule of $\frac{M}{A}$, hence L is nonzero submodule of M. But M is μ*- uniform , therefore $L \leq_{\mu^*e} M$. Since A is μ*- closed in M , then $\frac{L}{A} \leq_{\mu^*e} \frac{M}{A}$, by Prop. (2.12). Thus $\frac{M}{A}$ is μ*- uniform. □

A direct sum of μ*- uniform modules need not be μ*- uniform. For example , let $M = Z_8 \oplus Z_2$ as Z- module, clearly that Z_8 and Z_2 are μ*- uniform Z- modules but M is not μ*- uniform , where there exists a singular submodule $A = <(\bar{0},\bar{1})>$ which is not essential in M since there is $B = <(\bar{2},\bar{0})>$ such that $A \cap B = 0$.

Now , we give certain conditions under which a direct sum of μ*- uniform modules is μ*- uniform.

Let M be an R- module. Recall that a submodule A of M is called a **fully invariant** if $g(A) \leq A$, for every $g \in End(M)$ and M is called **duo module** if every submodule of M is fully invariant. See [8].

Proposition (2.26): Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are μ*- uniform modules , then M is μ*- uniform. Provided that $A \cap M_i \neq 0$, $\forall\, i = 1,2$.

Proof: Let A be a nonzero submodule of M. Since M is duo module , then A is fully invariant and hence $A = (A \cap M_1) \oplus (A \cap M_2)$. Since each of $(A \cap M_1)$ and $(A \cap M_2)$ is a nonzero submodule of M_1 and M_2 respectively , it follows that $(A \cap M_1) \leq_{\mu^*e} M_1$ and $(A \cap M_2) \leq_{\mu^*e} M_2$. Then $A \leq_{\mu^*e} M$, by Prop. (2.8). □

Recall that an R- module M is called **distributive** if for all A , B and $C \leq M$, $A \cap (B+C) = (A \cap B)+(A \cap C)$. See [9].

In similar argument one can easily prove the following proposition.

Proposition (2.27): Let $M = M_1 \oplus M_2$ be a distributive module. If M_1 and M_2 are μ*- uniform modules , then M is μ*- uniform. Provided that $A \cap M_i \neq 0$, $\forall\, i = 1,2$.

3. μ*-Extending modules.

In this section , we introduce the concept of μ*- extending modules as a generalization of extending modules. We generalize some properties of extending modules to μ*-

extending modules and discuss when the submodule of μ*-extending module is μ*- extending module.

Definition (3.1): An *R*- module *M* is called **μ*- extending module** if every submodule of *M* is μ*- essential in a direct summand. Clearly that every μ*- uniform module is μ*-extending. The converse is not true in general. For example , Z_6 as *Z*- module.

Remarks and Examples (3.2).

(1) Every extending module is μ*- extending , hence *Z* as *Z*- module is μ*- extending. The converse is not true in general . For example , let $R = Z[x]$ be a polynomial ring of integers *Z* and let $M = Z[x] \oplus Z[x]$. Since *M* is nonsingular , then it is μ*- uniform and hence it is μ*-extending , but *M* is not extending , by [2 , P.109].

(2) Let *M* be a singular *R*- module. Then *M* is μ*-extending if and only if *M* is extending.

(3) Let *M* be a torsion module over a commutative integral domain. Then *M* is μ*- extending if and only if *M* is extending.

(4) Let *M* be a prime *R*- module with $Z(M) \neq 0$. Then *M* is μ*- extending if and only if *M* is extending.

(5) For any prime number *p* , the *Z*- module $M = Z_p \oplus Z_{p2}$ is μ*- extending.

(6) For any prime number *p* , the *Z*- module $M = Z_p \oplus Z_{p3}$ is not μ*- extending.

The following proposition gives a condition under which the μ*- extending module and μ*- uniform module are equivalent.

Proposition (3.3): Let *M* be an indecomposable module. Then the following statements are equivalent.

(1) *M* is μ*- uniform.
(2) *M* is μ*- extending.
(3) Every cyclic submodule of *M* is μ*- essential in a direct summand of *M*.

Proof: $(1) \Rightarrow (2) \Rightarrow (3)$ It is clear.

$(3) \Rightarrow (1)$ Assume that every cyclic submodule of *M* is μ*-essential in a direct summand of *M* and let *A* be a nonzero submodule of *M* , let $x \in A$, hence *Rx* is μ*- essential in a direct summand *D* of *M*. But *M* is indecomposable, therefore $D = M$. Since $Rx \leq A \leq M$, then $A \leq_{\mu^*e} M$. Thus *M* is μ*-uniform. □

Now , we give various conditions under which a submodule of a μ*- extending module is μ*- extending.

Proposition (3.4): Let *M* be a μ*- extending *R*- module and let *A* be a submodule of *M* such that the intersection of *A* with any direct summand of *M* is a direct summand of *A*, then *A* is a μ*- extending module.

Proof: Let $X \leq A \leq M$. Since *M* is μ*- extending , then there exists a direct summand *D* of *M* such that $X \leq_{\mu^*e} D$. By our assumption $A \cap D$ is a direct summand of *A*. Hence $X = (X \cap A) \leq_{\mu^*e} (A \cap D)$, by Prop. (2.8). Thus *A* is μ*- extending.

□

Let *M* be an *R*- module. Recall that a submodule *A* of *M* is called a **fully invariant** if $g(A) \leq A$, for every $g \in End(M)$ and *M* is called **duo module** if every submodule of *M* is fully invariant. See [8].

Proposition (3.5): Every fully invariant submodule of μ*-extending module is μ*- extending.

Proof. Let *M* be a μ*- extending module and let *A* be a fully invariant submodule of *M*. Let *X* be a submodule of *A*. Since *M* is μ*- extending , then there exists a direct summand *D* of *M* such that $X \leq_{\mu^*e} D$. Let $M = D \oplus D'$, where $D' \leq M$. Now consider the projection map $p: M \longrightarrow D$,then $(1-p): M \longrightarrow D'$. Claim that $A=(A \cap p(M)) \oplus ((I-p)(M) \cap A)$. To show that , let $x \in A$, then $x = a+b$, $a \in D$ and $b \in D'$. Now $P(x) = p(a+b) = a$ and $(1-p)(x) = b$. But *A* is fully invariant , therefore $p(x) = a \in p(M) \cap A$ and $(1-p)(x) = b \in (1-p)(M) \cap A$. Thus $A=(A \cap p(M)) \oplus ((I-p)(M) \cap A) = (A \cap D) \oplus (A \cap D')$. Since $X \leq_{\mu^*e} D$, then $X = (X \cap A) \leq_{\mu^*e} (A \cap D)$.Thus *A* is μ*-extending , by Prop.(2.8). □

Corollary (3.6): Let *M* be a duo μ*- extending module , then every submodule of *M* is μ*- extending.

The next proposition gives another condition under which the submodule of μ*- extending module is a μ*- extending.

Recall that an *R*- module *M* is called **distributive** if for all *A* , *B* and $C \leq M$, $A \cap (B+C) = (A \cap B)+(A \cap C)$. See [9].

Proposition (3.7): Let *M* be a distributive μ*- extending *R*-module, then every submodule of *M* is μ*- extending.

Proof: Let *A* be a submodule of *M* and let *X* be a submodule of *A*. Since *M* is μ*- extending , then there exists a direct summand *D* of *M* such that $X \leq_{\mu^*e} D$, let $M=D \oplus D'$, where $D' \leq M$. But *M* is distributive, therefore $A=(A \cap D) \oplus (A \cap D')$, then $(A \cap D)$ is a direct summand of *A* and $X \leq_{\mu^*e} (A \cap D)$. Thus *A* is μ*-extending. □

Let *M* be an *R*- module. Recall that a proper submodule *A* of *M* is called a **maximal submodule** if whenever $A \subset B \leq M$, then $B = M$. Equivalently , *A* is maximal submodule if $M = Rx+A$, $\forall x \notin A$, see [10].

Proposition (3.8): Let M be a μ*- extending module which contains maximal submodules. Then for any maximal submodule A of M, either $A \leq_{\mu^*e} M$ or $M = A \oplus B$, for some simple submodule B of M.

Proof: Let A be a maximal submodule of M and suppose that A is not μ*- essential submodule of M, then there is a nonzero singular submodule B of M such that $A \cap B = 0$, let $x \in B$ and $x \notin A$. Since A is maximal submodule of M, then $M = A + Rx \leq A+B$, hence $M = A \oplus B$. Since $B \cong \frac{M}{A}$, so B is simple. □

A module M is called **local module** if it has a largest submodule, i.e, a proper submodule which contains all other proper submodules. For a local module M, Rad(M), the Jacobson radical of M is small in M, see [11].

Corollary (3.9): Let M be a local μ*- extending module, then $Rad(M) \leq_{\mu^*e} M$.

Proof: Since M is local module, then $Rad(M) << M$, hence $Rad(M)$ can not be a direct summand of M. Thus $Rad(M) \leq_{\mu^*e} M$, by Prop. (3.8). □

4. Characterizations of μ*-extending modules.

In this section, we give various characterizations of μ*- extending modules. Also, we give some conditions under which the direct sum of μ*- extending modules is μ*- extending module.

Theorem (4.1): Let M be an R- module. Then M is μ*- extending module if and only if every μ*- closed submodule of M is a direct summand.

Proof: ($\Rightarrow$) Suppose that M is μ*- extending and let A be a μ*- closed in M, then there is a direct summand D of M such that $A \leq_{\mu^*e} D$. But A is μ*- closed in M, therefore $A = D$.

($\Leftarrow$) To show that M is μ*- extending, let A be a submodule of M, then there is a μ*- closed submodule B of M such that $A \leq_{\mu^*e} B$, by Prop. (2.10). By our assumption, B is a direct summand of M. Thus M is μ*- extending module. □

Theorem (4.2): Let M be an R- module. Then the following statements are equivalent.

(1) M is μ*- extending module.

(2) For every submodule A of M, there is a decomposition $M = D \oplus D'$, such that $A \leq D$ and $D'+A \leq_{\mu^*e} M$.

(3) For every submodule A of M, there is a decomposition $\frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A}$ such that D is a direct summand of M and $K \leq_{\mu^*e} M$.

Proof: (1)$\Rightarrow$(2) Let M be a μ*- extending and let A be a submodule of M, there is a direct summand D of M such that $A \leq_{\mu^*e} D$, then $M = D \oplus D'$, $D' \leq M$. Since $\{A, D'\}$ is an independent family, then $A+D' \leq_{\mu^*e} M$, by Prop. (2.8).

(2)$\Rightarrow$(3) Let A be a submodule of M. By (2), there is a decomposition $M = D \oplus D'$, such that $A \leq D$ and $D'+A \leq_{\mu^*e} M$. Claim that $\frac{M}{A} = \frac{D}{A} \oplus \frac{D'+A}{A}$. Since $M = D \oplus D'$, then $\frac{M}{A} = \frac{D+D'}{A} = \frac{D}{A} + \frac{D'+A}{A}$ and $\frac{D}{A} \cap \frac{D'+A}{A} = \frac{D \cap (D'+A)}{A} = \frac{A+(D \cap D')}{A} = A$, hence $\frac{M}{A} = \frac{D}{A} \oplus \frac{D'+A}{A}$. Take $K = D'+A$, so we get the result.

(3)$\Rightarrow$(1) To show that M is μ*- extending, let A be a submodule of M. By (3), there is a decomposition $\frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A}$ such that D is a direct summand of M and $K \leq_{\mu^*e} M$. It is enough to show that $A \leq_{\mu^*e} D$. Let $i : D \rightarrow M$ be the injection map. Since $K \leq_{\mu^*e} M$, then $i^{-1}(K) \leq_{\mu^*e} i^{-1}(M)$, that is $D \cap K \leq_{\mu^*e} D$. One can easily show that $D \cap K = A$, so M is μ*- extending module. □

Proposition (4.3): Let M be an R- module. Then M is μ*- extending module if and only if for each μ*- closed submodule A of M, there is a complement B of A in M such that every homomorphism $f : A \oplus B \rightarrow M$ can be lifted to a homomorphism $g : M \rightarrow M$.

Proof: This is a direct consequence of [12, Lemma 2]. □

Proposition (4.4): Let M be an R- module. Then M is μ*- extending module if and only if for every submodule A of M, there exists an idempotent $f \in$ End (M) such that $A \leq_{\mu^*e} f(M)$.

Proof: Clear.

The following proposition gives another characterization of μ*- extending module.

Proposition (4.5): Let M be an R- module, then M is μ*- extending module if and only if for each direct summand A of the injective hull $E(M)$ of M, there exists a direct summand D of M such that $(A \cap M) \leq_{\mu^*e} D$.

Proof: Let A be a submodule of M and let B be a complement of A, then $A \oplus B \leq_e M$, by [1, Prop. (1.3)]. Since $M \leq_e E(M)$, then $A \oplus B \leq_e E(M)$. Thus $E(A) \oplus E(B) =$

$E(A \oplus B) = E(M)$. By our assumption , there exists a direct summand D of M such that $E(A) \cap M \leq_{\mu^*e} D$. But $A \leq_e E(A)$, therefore $A \cap M \leq_{\mu^*e} E(A) \cap M \leq_{\mu^*e} D$, hence $A \leq_{\mu^*e} D$. Thus M is μ*- extending. The proof of the converse is clear. □

The following proposition shows that the direct summand of μ*- extending module is μ*- extending.

Proposition (4.6): A direct summand of μ*- extending module is μ*- extending.

Proof: Let $M = A \oplus B$ be a μ*- extending module. To show that A is a μ*- extending , let X be a μ*- closed submodule of A , then $X \oplus B$ is a μ*- closed submodule of M , by Prop. (2.19). Hence $X \oplus B$ is a direct summand of M , then $M = X \oplus B \oplus Y$, $Y \leq M$, that is X is a direct summand of M. But $X \leq A$, therefore X is a direct summand of A. Thus A is μ*- extending module. □

The following proposition gives a condition under which a quotient of μ*- extending module is a μ*- extending.

Proposition (4.7): Let M be a μ*- extending module and let A be a μ*- closed submodule of M , then $\frac{M}{A}$ is μ*- extending module.

Proof: Let M be a μ*- extending module and let A be a μ*- closed submodule of M , then A is a direct summand of M , let $M = A \oplus A'$, for some submodule A' of M , hence $\frac{M}{A} \cong A'$ is a μ*- extending module , by Prop. (3.6). □

Corollary (4.8): Assume that $f : M \rightarrow M'$ is an R-homomorphism and let $Kerf$ be a μ*- closed submodule of M , then $f(M)$ is μ*- extending.

Proof: Let $f : M \rightarrow M'$ be an R- homomorphism and let $Kerf$ be a μ*- closed submodule of M , then $\frac{M}{Kerf} \cong f(M)$ is μ*- extending module. □

The direct sum of μ*- extending modules need not be μ*- extending , for example , let $M = Z_8 \oplus Z_2$ as Z- module, clearly that Z_8 and Z_2 are μ*- extending Z- module but M is not μ*- extending.

Now , we give sufficient conditions under which the direct sum of μ*-extending modules is a μ*-extending.

Proposition (4.9): Let $M=M_1 \oplus M_2$ be a distributive module if M_1 and M_2 are μ*-extending , then M is μ*- extending.

Proof: Let $M = M_1 \oplus M_2$ be a distributive module , M_1 and M_2 are μ*-extending and let $A \leq M$. Since M is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 , M_2 are μ*-extending , then there exists a direct summand D_1 of M_1 and direct summand D_2 of M_2 such that $(A \cap M_1) \leq_{\mu^*e} D_1$ and $(A \cap M_2) \leq_{\mu^*e} D_2$. Hence $A = (A \cap M_1) \oplus (A \cap M_2)) \leq_{\mu^*e} (D_1 \oplus D_2)$ and $D_1 \oplus D_2$ is a direct summand of M , by Prop. (2.8). Thus M is μ*- extending. □

Proposition (4.10): Let $M = \bigoplus_{i \in I} M_i$ be an R-module ,where M_i is a submodule of M ,∀ $i \in I$. If M_i is μ*-extending , for each $i \in I$ and every μ*- closed submodule of M is fully invariant , then M is μ*-extending.

Proof: Let A be a μ*- closed submodule of M and $\pi_i : M \longrightarrow M_i$ be the natural projection on M_i , for each $i \in I$. Let $x \in A$, then $x = \sum x_i$, $i \in I$, $x_i \in M_i$, $\pi_i(x) = x_i$. By our assumption, A is fully invariant and hence $\pi_i(A) \leq A \cap M_i$. So, $\pi_i(x) = x_i \in A \cap M_i$ and hence $x \in \bigoplus_{i \in I} (A \cap M_i)$. Thus $A \leq \bigoplus_{i \in I} (A \cap M_i)$. But $\bigoplus_{i \in I} (A \cap M_i) \leq A$, therefore $A = \bigoplus_{i \in I} (A \cap M_i)$, $\forall\, i \in I$ .Since $A \cap M_i \leq M_i$ and M_i is μ*-extending , then there exists direct summands D_i of M_i such that $(A \cap M_i) \leq_{\mu^*e} D_i$. By Prop. (2.8) $A = (\bigoplus_{i \in I} (A \cap M_i)) \leq_{\mu^*e} (\bigoplus_{i \in I} D_i)$, for each $i \in I$. Thus M is μ*-extending. □

Proposition (4.11) Let M_1 and M_2 be μ*-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is μ*-extending.

Proof: Let A be a submodule of $M_1 \oplus M_2$. Since $annM_1 + annM_2 = R$, then by the same way of the proof of [13 ,Prop.4.2,CH.1] $A = B \oplus C$, where B is a submodule of M_1 and C is a submodule of M_2. Since M_1 and M_2 are μ*-extending , then there exists direct summands D_1 of M_1 and D_2 of M_2 such that $B \leq_{\mu^*e} D_1$ and $C \leq_{\mu^*e} D_2$, hence $A = (B \oplus C) \leq_{\mu^*e} (D_1 \oplus D_2)$, by Prop. (2.8). Thus M is μ*- extending. □

Proposition (4.12): Let $M = M_1 \oplus M_2$ be an R- module with M_1 being μ*- extending and M_2 is semisimple. Suppose that for any submodule A of M with $A \cap M_1$ is a direct summand of A. Then M is μ*- extending.

Proof: Let A be a submodule of M. Then it is easy to see that $A+M_1 = M_1 \oplus [(A+M_1) \cap M_2]$. Since M_2 is semisimple , then $(A+M_1)\cap M_2$ is a direct summand of M_2 and therefore $A+M_1$ is a direct summand of M. By our assumption $A = (A \cap M_1)\oplus A'$, for some submodule A' of M. Since M_1 is μ*-extending , then there is a direct summand D of M_1 such that $A\cap M_1 \leq_{\mu^*e} D$. Hence $A = (A\cap M_1)\oplus A' \leq_{\mu^*e} D\oplus A'$. Since $D\oplus A' \leq_{\oplus} A+M_1 \leq_{\oplus} M$, then $D\oplus A'$ is a direct summand of M. Thus M is μ*- extending. □

Proposition (4.13): Let $M = M_1\oplus M_2$ with M_1 being μ*-extending and M_2 injective. Suppose that for any submodule A of M , we have $A\cap M_2$ is a direct summand of A, then M is μ*- extending.

Proof: Let A be a submodule of M. By hypothesis , there is a submodule A' of A such that $A = (A\cap M_2)\oplus A'$. Note that $A' \cap M_2 = 0$ and hence $\frac{M_2 + A'}{A'} \cong M_2$ is an injective module , so there is a submodule M' of M such that $\frac{M}{A'} = \frac{M_2 + A'}{A'} \oplus \frac{M'}{A'}$. Thus it is easy to see that $M = M_2\oplus M'$ and that $M' \cong \frac{M}{M_2} \cong M_1$. Since M_1 is μ*- extending , then M' is μ*- extending , there is a direct summand K of M' such that $M = K\oplus K'$ and $A' \leq_{\mu^*e} K$. Since $A\cap M_2$ is a submodule of M_2 and M_2 is an injective module , then there is a direct summand D of M_2 such that $A\cap M_2 \leq_{\mu^*e} D$. Hence $A = [(A\cap M_2)\oplus A'] \leq_{\mu^*e} D\oplus K$, where $D\oplus K$ is a direct summand of M. Thus M is μ*- extending. □

Proposition (4.14): Let $M = M_1\oplus M_2$ such that M_1 is μ*-extending and M_2 is injective module. Then M is μ*-extending module if and only if for every submodule A of M such that $A\cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{\mu^*e} D$.

Proof: Suppose that for every submodule A of M such that $A \cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{\mu^*e} D$. Let A be a submodule of M such that $A\cap M_2 = 0$. Since $\frac{M_2 + A}{A} \cong M_2$ is an injective module , there is a submodule M' of M containing A such that $\frac{M}{A} = \frac{M'}{A} \oplus \frac{(M_2 + A)}{A}$. It is easy to see that $M = M' \oplus M_2$. Since $M' \cong \frac{M}{M_2} \cong M_1$ is μ*- extending , so there is a direct summand K of M' , hence K is a direct summand of M , such that $A \leq_{\mu^*e} K$. Thus M is μ*- extending. The proof of the converse is obvious. □

Proposition (4.15): Let R be a PID , then the following statements are equivalent:

1- $\bigoplus_I R$ is μ*-extending, for every index set I.

2- Every projective R- module is μ*-extending.

Proof: (1)⇒(2) Let M be a projective R- module , then by [10, Corollary (4.4.4), p.89] ,there exists a free R- module F and an epimorphism $f : F \longrightarrow M$. Since F is free, then $F \cong \bigoplus_I R$, for some index set I. Now consider the following short exact sequence:

$$0 \longrightarrow Kerf \xrightarrow{\ i\ } \bigoplus_I R \xrightarrow{\ f\ } M \longrightarrow 0$$

Where i is the inclusion map . Since M is projective , then the sequence splits .Thus $\bigoplus_I R = Kerf \oplus M$. Since $\bigoplus_I R$ is μ*-extending , then M is μ*- extending , by Prop. (4.6).

(2)⇒(1) Clear. □

By the same argument ,we can prove the following:

Proposition(4.16): Let R be a PID , then the following statements are equivalent:

1- $\bigoplus_I$ R is μ*-extending, for every finite index set I.

2- Every finitely generated projective R- module is μ*-extending.

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SEMI-T-ABSO FUZZY SUBMODULES AND SEMI-T-ABSO FUZZY MODULES

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Abstract. Let Ḿ be a unitary R-module and R be a commutative ring with identity and let X be a fuzzy module of an R-module Ḿ. Our aim in this paper to study the concepts semi T-ABSO fuzzy submodules and semi T-ABSO fuzzy modulesas generalizations of T-ABSO fuzzy submodules and T-ABSO fuzzy modules. Many new basic properties, characterizations and relationships between semi T-ABSO fuzzy submodules(modules) and other concepts are given.

Keywords. T-ABSO fuzzy submodule,T-ABSO fuzzy module, semi T-ABSO fuzzy ideal, semi T-ABSO fuzzy submodule, semi T-ABSO fuzzy module, quasi-prime fuzzy submodule, semiprime fuzzy submodule.

1. Introduction

Zahedi [17], in1992 presented the concept of a fuzzy ideal A fuzzy subset K of a ring R is called a fuzzy ideal of R, if $\forall\, x, y \in$ R: $K(x\text{-}y) \geq \min\{K(x), K(y)\}$ and $K(xy) \geq \max\{K(x),K(y)\}$". Mukhrjee [13], in 1989 intoduced the concept of prime fuzzy ideal " A fuzzy ideal Ĥ of a ring R is called a prime fuzzy ideal if Ĥ is a non-empty and for all a_s, b_l fuzzy singletons of R such that $a_s b_l \subseteq$ Ĥ implies that either $a_s \subseteq$ Ĥ or $b_l \subseteq$ Ĥ, $\forall s,l \in [0,1]$".Deniz et al [3], in 2017 presented the concept of 2-absorbing fuzzy ideal which is a generalization of prime fuzzy ideal. Darani and Soheilnia [2], in 2011 introduced the concept of 2-absorbing submodule "a proper submodule N of Ḿ is called 2-absorbing submodule of Ḿ if whenever a , b ∈ R , m∈ Ḿ and abm ∈N , then am ∈N or bm ∈N or ab $\in(N{:}_R \acute{M})$". Hatam and wafaa [7], in 2018 expanded this concept " Let X be fuzzy module of an R-module Ḿ. A proper fuzzy submodule A of X is called T-ABSO fuzzy submodule if whenever a_s, b_l be fuzzy singletons of R, and $x_v \subseteq X$, $\forall\, s, l, v \in [0,1]$ such that $a_s b_l x_v \subseteq A$ then either $a_s b_l \subseteq (A{:}_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$" Abdulrahman [1], in 2015 presented the definition of 2-absorbing module" An R-module Ḿ is called a 2-absorbing module if zero (0) submodule of Ḿ is 2-absorbing submodule " equivalently " if whenever a, b ∈ R , m∈ Ḿ and abm =0 , then am =0 or bm =0 or ab ∈ annḾ". Hadi [4], in 2004 presented the concept of semiprime fuzzy submodules "Let A be a fuzzy submodule of a fuzzy module X of an R-module Ḿ such that A≠ X, A is called semiprime fuzzy submodule if for each fuzzy singletone r_k of R, $x_v \subseteq X$, $r_k^2 x_v \subseteq A$ implies $r_k x_v \subseteq A$ " Maysoun [11], in 2012 introduced the concept of semiprime fuzzy module "Let X be a fuzzy module of an R-module Ḿ, X is called semiprime fuzzy module if for each non-empty fuzzy submodule A of X, F-annA is a semiprime fuzzy ideal of R".Hatam [6], in 2001 introduced the concept of quasi-prime fuzzy submodule" A fuzzy submodule A of a fuzzy module X of an R-module M is called a quasi-prime fuzzy submodule of X if whenever $a_s b_l x_v \subseteq A$ for fuzzy singletons a_s, b_l of R and $x_v \subseteq X$, $\forall s,l,v \in$ L, implies that $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$".Also Abdulrahman [1], in 2015 is circulated the concepts of 2-absorbing submodules and 2-absorbing modules to semi-2-absorbing submodules and semi-2-absorbing modules.

This paper be composed of two sections

In section (1) we present and study the concept of semi T-ABSO fuzzy submodule as a generalization of T-ABSO fuzzy submodule and we give many properties, characterizations and relationships between semi T-ABSO fuzzy and other concepts.
Futhermore we debate the direct sum of semi T-ABSO fuzzy submodules. In section(2) we present the concept of semi T-ABSO fuzzy modules , so many properties and characterizations are given . Also we debate the direct sum of semi T-ABSO fuzzy modules.
Note that we denote to fuzzy: F., module: M., submodule: subm. , [0,1]: L , otheroiwse: o.w.

2. Semi T-ABSO F. Subm.

In this section we present the concepts of semi-T-ABSO F. ideal and semi T-ABSO F. subm. Also introduced and study some properties and relations of semi-T F. subm. with other concepts of F. subm.

Frist we give the proposition specificates of T-ABSO F. subm. in terms of its level subm. is given:

"**Proposition 2.1.** Let A be T-ABSO F. subm. of F. M. X of an R- M. Ḿ iff the level subm. A_v is T-ABSO subm. of X_v, for all $v \in$ L, [7]".

Now, we present the concepts of a semi T-ABSO F. ideal.and semi T-ABSO F. subm. as follows:

Definition 2.2. A proper F. ideal Ĥ of a ring R is called a semi T-ABSO F. ideal if for F. singletons a_s, b_l of R such that $a_s^2 b_l \subseteq$ Ĥ , $\forall\, s, l \in$ L, implies either $a_s b_l \subseteq$ Ĥ or $a_s^2 \subseteq$ Ĥ ; that is Ĥ a semi T-ABSO F. subm. of X of an R- M. R.

Definition 2.3. A proper F. subm. A of F. M. X of an R- M. Ḿ is called a semi T-ABSO F. subm. of X if for F. singletons a_s of R and $x_v \subseteq X$ such that $a_s^2 x_v \subseteq A$, $\forall s,v \in$ L, implies either $a_s x_v \subseteq A$ or $a_s^2 \subseteq (A{:}_R X)$.

The proposition specificates a semi T-ABSO F. subm. in terms of its level subm is given:

Proposition 2.4. Let A be F. subm. of F. M. X of an R- M. Ḿ. Then A is a semi T-ABSO F. subm. of X iff the level A_v is a semi T-ABSO subm. of X_v, ∀ v∈ L.

Proof. (⟹) Let $a^2x \in A_v$ for each a∈ R , $x \in X_v$, ∀ v∈ L, then $A(a^2x) \geq v$, hence $(a^2x)_v \subseteq A$ so that $a_s^2 x_k \subseteq A$ where v=min{s, k} and $(a^2)_s = a_s^2$. But A is a semi-T-ABSO F. subm., then either $a_s x_k \subseteq A$ or $a_s^2 \subseteq (A:_R X)$, hence $(ax)_v \subseteq A$ or $(a^2)_v \subseteq (A:_R X)$, implies $ax \in A_v$ or $a^2 \in (A_v:_R X_v)$. Thus A_v is a semi-T-ABSO of X_v.

(⟸) Let $a_s^2 x_k \subseteq A$ for F. singleton a_s of R and $x_v \subseteq X$, ∀ s,k ∈ L, then $(a^2x)_v \subseteq A$ where v=min{s, k}, hence $A(a^2x) \geq v$ so that $a^2x \in A_v$. But A_v is a semi T-ABSO subm. of X_v, then either $ax \in A_v$ or $a^2 \in (A_v:_R X_v)$, hence $(ax)_v \subseteq A$ or $(a^2)_v \subseteq (A:_R X)$, so that $a_s x_k \subseteq A$ or $a_s^2 \subseteq (A:_R X)$. Thus A is a semi T-ABSO F. subm. of X.

Remarks and Examples 2.5

(1) Every semiprime F. subm. is a semi T-ABSO F. subm.

Proof:

Let $a_s^2 x_v \subseteq A$ for F. singleton a_s of R and $x_v \subseteq X$. Since semiprime F. subm., then that $a_s x_v \subseteq A$. So that *A* is a semi T-ABSO F. subm.

However the converse incorrect, for example:

Let X:Z→L such that $X(y)=\begin{cases}1 & if\ y \in Z\\ 0 & o.w.\end{cases}$

It is obvious that X is F. M. of *Z*- M. Z.

Let *A*: Z→L such that $A(y)=\begin{cases}\frac{1}{2} & if\ y \in 4Z\\ 0 & o.w.\end{cases}$

It is obvious that *A* is a fuzzy submodule of *X*.

Now, *A* is a semi T-ABSO fuzzy submodule of *X* since $2^2_{\frac{1}{3}}.1_{\frac{1}{3}}=4_{\frac{1}{3}} \subseteq A$, $2^2_{\frac{1}{3}} = 4_{\frac{1}{3}} \subseteq A$ where $A(4)=\frac{1}{2} > \frac{1}{3}$, but *A* is not semiprime fuzzy submodule since $2_{\frac{1}{3}}.\ 1_{\frac{1}{3}}=2_{\frac{1}{3}} \not\subseteq A$ because $A(2)=0 \ngtr \frac{1}{3}$.

(2) It obvious that every T-ABSO F. subm. is semi T-ABSO F. subm. However the convrse incorrect for example:

Let X:Z⊕Z→L such that $X(x,y)=\begin{cases}1 & if\ (x,y) \in Z\oplus Z\\ 0 & o.w.\end{cases}$

It is obvious that X is F. M. of *Z*- M. Z⊕Z.

Let *A*: Z⊕Z →L such that

$A(x,y)=\begin{cases}v & if\ (x,y) \in 10Z\oplus(0)\\ 0 & o.w.\end{cases}\ \forall v \in L$

It is obvious that A is F. subm. of X .

Now, A_v=10Z⊕(0) is not T-ABSO subm. in X_v=Z⊕Z as Z-M. since 2.5(1,0)=(10,0)∈10Z⊕(0), but 2(1,0)∉10Z⊕(0), 5(1,0)∉10Z⊕(0) and 2.5∉(10Z⊕(0):$_Z$ Z⊕Z)=(0). But A_v is a semi T-ABSO subm. since if $r^2(x,0) \in A_v$ then $r^2x \in 10Z$, hence it obvious that 10Z is semiprime, that is r x∈10Z, Thus r(x,0)∈10Z⊕(0)= A_v. Then A_v is a semi T-ABSO subm. Thus *A* is a semi T-ABSO F. subm.

(3) Every a quasi-prime F. subm. is a semi T-ABSO F. subm. However the converse incorrect. Consider the example in part(1) where *A* is semi T- ABSO F. subm., but *A* is not quasi-prime F. since $2_{\frac{1}{3}}.\ 2_{\frac{1}{3}}.\ 1_{\frac{1}{3}}=4_{\frac{1}{3}} \subseteq A$, but $2_{\frac{1}{3}}.\ 1_{\frac{1}{3}}=2_{\frac{1}{3}} \not\subseteq A$.

(4) Let *A* , *B* be F. subm. of F. M. *X* of an R- M. Ḿ and *A*⊆*B*. If *A* is a semi T-ABSO F. subm. of *X* then *A* is a semi T-ABSO F. subm. of *B*.

Proof. Let be F. singleton r_k of R and $x_v \subseteq B$ such that $r_k^2 x_v \subseteq A$, ∀k,v ∈ L. Since *B* is F. subm. of X then $x_v \subseteq X$ and $r_k^2 x_v \subseteq A$, then either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. If $r_k^2 \subseteq (A:_R X)$ then $r_k^2 X \subseteq A$ and since *B* is F. subm. of *X*, hence $r_k^2 B \subseteq r_k^2 X$, so that $r_k^2 B \subseteq A$ implies $r_k^2 \subseteq (A:_R B)$. Thus *A* is a semi T-ABSO F. subm of *B*.

(5) The intersection of two semi T-ABSO F. subms is not necessary that a semi T- ABSO F. subm., for example:

Let X: Z_{12}→L such that $X(y)=\begin{cases}1 & if\ y \in Z_{12}\\ 0 & o.w.\end{cases}$

It is clear that X is F. M. of *Z*- M. Z.

Let *A*: Z_{12}→L such that $A(y)=\begin{cases}v & if\ y \in \overline{(4)}\\ 0 & o.w.\end{cases}\ \forall v \in L$

Let *B*: Z_{12}→L such that $B(y)=\begin{cases}v & if\ y \in \overline{(6)}\\ 0 & o.w.\end{cases}\ \forall v \in L$

It is obvious that A and B are F. subms of X .

Now, $A_v=\overline{(4)}$, $B_v=\overline{(6)}$ and $X_v=Z_{12}$ as Z- M. It is obvious that A_v and B_v are semi T-ABSO subms, but $A_v \cap B_v = \overline{(4)} \cap \overline{(6)} = \overline{(0)}$ is not semi T-ABSO subm. since $2^2.\ \overline{(3)} = \overline{(0)}$,but 2. $\overline{(3)} \neq \overline{(0)}$and $2^2 \notin annZ_{12} = 12Z$. So that *A* and *B* are semi T-ABSO F. subms, but *A*∩*B* is not a semi T-ABSO F. subm. of *X*.

(6) Let X:Z→L such that $X(y)=\begin{cases}1 & if\ y \in Z\\ 0 & o.w.\end{cases}$

It is obvious that X be F. M. of *Z*- M. Z.

Let *A*: Z→L such that $A(y)=\begin{cases}v & if\ y \in p^2Z\\ 0 & o.w.\end{cases}\ \forall v \in L$

Where p is a prime number.

It is obvious that A is F. subm. of X.

Now, $A_v = p^2Z$ and $X_v = Z$ as Z- M

It is obvious that A_v , p is prime number is a semi T-ABSO subm. Thus *A* is a semi T-ABSO F. subm. of *X*.

(7) Let *A*, *B* be two F. subm. of F. M. *X* of an R- M. Ḿ suth that *A*≅*B*. If *A* is a semi T- ABSO F. subm. then it is not necessary that B is a semi T-ABSO F. subm.for example

Let X:Z→L such that $X(y)=\begin{cases}1 & if\ y \in Z\\ 0 & o.w.\end{cases}$

It is obvious that *X* is F. M. of *Z*- M. Z.

Let *A*: Z→L such that $A(y)=\begin{cases}v & if\ y \in 4Z\\ 0 & o.w.\end{cases}\ \forall v \in L$

Let *B*: Z→L such that $B(y)=\begin{cases}v & if\ y \in 60Z\\ 0 & o.w.\end{cases}\ \forall v \in L$

It is obvious that A and B are F. subm. of X.

Now, A_v=4Z , B_v=60Z are subm.of X_v=Z as Z- M. and 4Z≅60Z , but $A_v = 4Z$ is semi T-ABSO while B_v=60Z is not semi T-ABSO subm. of X_v. So that *A*≅*B* where *A* is a semi T-ABSO F. subm., but *B* is not semi T-ABSO F. subm of *X*.

(8) If *A* is semi T-ABSO F. subm. of F. M. *X* of an R- M. Ḿ and *B*⊆*A*, it may be that *B* is not semi T-ABSO F. subm. for example:

Consider the example in part(7), where *A* is a semi T-ABSO F. subm., *B*⊂*A* since $B_v = 60Z \subset A_v = 4Z$, but *B* is not semi T-ABSO F. subm. of *X*.

Recall that "Let *A* be a F. subm. of F. M. *X* of an R-module Ḿ, then *A* is called an irreducible F. subm. if for all two F. subms *B* and *K* such that *B*∩*K*=*A* then *B*=*A* or *K*=*A* otherwise *A* is called reducible, [12]".

Proposition 2.6. Let *X* be F. M. of an R- M. Ḿ and *A* is irreducible F. subm. of *X*. Then the following expressions are equivalent:

1- *A* is T-ABSO F. subm. and $(A:_R X)$ is semi prime F. ideal.
2- *A* is a prime F. subm.
3- *A* is a semi prime F. subm.
4- *A* is a quasi prime F. subm.

5- A is T-ABSO F. subm. and $(A:_R X)$ is a prime F. ideal.

Proof. (1)⟹(2) Let $r_k(r_k x_v) \subseteq A$ for F. singleton r_k of R and $x_v \subseteq X$. Since A is T-ABSO F. subm., then $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. If $r_k x_v \subseteq A$ then we are done. If $r_k^2 \subseteq (A:_R X)$, then $r_k \subseteq (A:_R X)$ since $(A:_R X)$ is a semiprime F. ideal. so that A is a prime F. subm.

(1)⟹(3) Let $r_k^2 x_v \subseteq A$ for F. singletons r_k of R and $x_v \subseteq X$. Since A is T-ABSO F. subm., then $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. If $r_k x_v \subseteq A$ the proof is complete.

If $r_k^2 \subseteq (A:_R X)$, then $r_k \subseteq (A:_R X)$ since $(A:_R X)$ is a semi prime F. ideal. Hence $r_k x_v \subseteq A$. Thus A is a semi prime F. subm.

(2)⟹ (3)By [12].

(3)⟹ (4) By [6].

(4)⟹(5) Since A is a quasi prime F. subm., then A is T-ABSO F. subm. and $(A:_R X)$ is a prime F. ideal by [6].

(5)⟹(1) It is clear.

Proposition 2.7. Let X be F. M. of an R- M. $\acute{M}$ and A and B be F. subm. of X. Then A is a semi T-ABSO F. subm. iff $r_k^2 B \subseteq A$ for F. singleton r_k of R, ∀k∈ L, implies $r_k B \subseteq A$ or $r_k^2 \subseteq (A:_R X)$.

Proof. (⟹) Let $r_k^2 B \subseteq A$ for F. singleton r_k of R. Assume there exists $x_v \subseteq B$ such that $r_k x_v \nsubseteq A$, since $r_k^2 B \subseteq A$, hence $r_k^2 x_v \subseteq A$, but A is a semi T-ABSO F. subm. and $r_k x_v \nsubseteq A$.

Then $r_k^2 \subseteq (A:_R X)$.

(⟸) It is obvious.

Proposition 2.8. Let A be a proper F. subm. of F. M. of an R- M. Ḿ . If A is a semi T-ABSO F. subm. of X, then $(A:_R X)$ is a semi T-ABSO F. ideal.

Proof. Let a_s, b_l be F. singletons of R, such that $a_s^2 b_l \subseteq (A:_R X)$, hence $a_s^2 b_l X \subseteq A$, then $a_s^2 b_l x_v \subseteq A$ for each F. singleton $x_v \subseteq X$ and suppose that $a_s^2 \nsubseteq (A:_R X)$. Since A is a semi T-ABSO F. subm., hence $a_s b_l x_v \subseteq A$. So that $a_s b_l \subseteq (A:_R X)$. Then $(A:_R X)$ is semi T-ABSO F. ideal.

Recall that " A fuzzy module X of an R-module M is called a multiplication fuzzy module if for each non-empty fuzzy submodule A of X there exists a fuzzy ideal Ĥ of R such that A=ĤX,[6]".

The converse of Proposition (2.8) hold under the class of multiplication F. M. as follows:

Proposition 2.9. Let A be a proper F. subm. of a multiplication F. M. X of an R- M. Ḿ. If $(A:_R X)$ is a semi T-ABSO F. ideal, then A is a semi T-ABSO F. subm.

Proof. Let $a_s^2 x_v \subseteq A$ for F. singletons a_s of R and $x_v \subseteq X$.

Then $a_s^2 < x_v > \subseteq A$. But $< x_v > =$ ĤX for some F. ideal Ĥ of R. Since X is a multiplication F. M., then a_s^2 Ĥ $\subseteq (A:_R X)$. But $(A:_R X)$ is a semi T-ABSO F. ideal, then either a_sĤ $\subseteq (A:_R X)$ or $a_s^2 \subseteq (A:_R X)$ by Proposition (2.7). Then a_sĤ$X \subseteq A$ or $a_s^2 \subseteq (A:_R X)$. Thus $a_s < x_v > \subseteq A$ or $a_s^2 \subseteq (A:_R X)$. Then A is a semi T-ABSO F. subm.

Recall that "A F. M. X of an R-M $\acute{M}$ is called a cyclic F. M. if there exists $x_v \subseteq X$ such that $y_k \subseteq X$ written as $y_k = r_l x_v$ for some F. singleton r_l of R, where k,l,v ∈ L in this case, write X=$< x_v >$ to denote the cyclic F. M. generated by x_v, [6]".

Corollary 2.10. Let A be F. subm. of cyclic F. M. X of an R- M. Ḿ. Then A is a semi T-ABSO F. subm. iff $(A:_R X)$ is a semi T-ABSO ideal.

Proof. Since every cyclic F. M. is a multiplication F. M. by[6].

By Proposition (2.8) and Proposition (2.9), then the outcome is obtained.

Recall that "If X is F. M. of an R-M.$\acute{M}$, then X is called a finitely generated F. M. if there exists $x_1, x_2, x_3, ... \subseteq X$ such that X=$\{a_1(x_1)_{v_1} + a_2(x_2)_{v_2} + \cdots + a_n(x_n)_{v_n}\}$, where $a_i \in R$ and $a(x)_v = (ax)_v$, ∀v∈L. Where $(ax)_v(y) = \begin{cases} v & if\ y = ax \\ 0 & o.w. \end{cases}$, [8]".

Recall that "If X is F. M. of an R-M. $\acute{M}$, then X is said to be a faithful F. M. if $F\text{-}annX \subseteq 0_1$ where $F\text{-}annX = \{x_v : r_k x_v = 0_1\ \forall\, x_v \subseteq X$ and r_k is F. singleton of R$, \forall\, v,k \in L\}$, [15]".

Corollary 2.11. Let X be a faithful finitely generated multiplication F. M. of an R- M. Ḿ and A is a proper subm. of X. Then the following expressions are equivalent:

1- A is a semi T-ABSO F. subm. of X;

2- $(A:_R X)$ is a semi T-ABSO F. ideal;

3- A=ĤX for some semi T-ABSO F. ideal Ĥ of R.

Proof. (1)⟹(2) By Proposition (2.8).

(2)⟹(3) By [6, Proposition (2.2.2)], we get the result.

(3)⟹(1) Let $r_h^2 x_v \subseteq A$ for F. singleton r_h of R and $x_v \subseteq X$, then $r_h^2 < x_v > \subseteq A$. Since X is a multiplication F. M., so that $< x_v > = KX$ for some F. ideal K of R, then $r_h^2 KX \subseteq$ ĤX. Since X is a faithful finitely generated multiplication F. M., hence $r_h^2 K \subseteq$ Ĥ. But Ĥ is a semi T-ABSO F. ideal, so that either $r_h K \subseteq$ Ĥ or $r_h^2 \subseteq$ (Ĥ$:_R \lambda_R$) by Proposition (2.7). Hence $r_h KX \subseteq$ Ĥ$X = A$ or $r_h^2 \subseteq$ Ĥ $= ($Ĥ$X:_R X) = (A:_R X)$. Then $r_h x_v \subseteq A$ or $r_h^2 \subseteq (A:_R X)$.

Proposition 2.12. Let A be a proper F. subm. of F. M. X of an R- M. $\acute{M}$. Then the following expressions are equivalent:

1- A is a semi T-ABSO F. subm. of X;

2- $(A:_X$ Ĥ$)$ is a semi T-ABSO F. subm. for each F. ideal Ĥ of R such that Ĥ$X \nsubseteq A$;

3- $(A:_X < a_s >)$ is a semi T-ABSO F. subm. for each F. singleton a_s of R, $a_s X \nsubseteq A$.

Proof. (1)⟹(2) Since ĤX⊈A, hence $(A:_X$ Ĥ$) \neq X$ **.** Let $r_k^2 x_v \subseteq (A:_X$ Ĥ$)$ for F. singletons r_k of R , $x_v \subseteq X$**.** Thus r_k^2Ĥ$x_v \subseteq A$**.** By Proposition (2.7), either $r_k I x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$, hence $r_k x_v \subseteq (A:_X$ Ĥ$)$ or $r_k^2 \subseteq ((A:_X$ Ĥ$):_R X)$.

(2)⟹(3) It is obvious.

(3)⟹(1) Since $1_v X \nsubseteq A$, hence $(A:_R < 1_v >)$ is a semi T-ABSO F. subm., then A is a semi T-ABSO F. subm. since $(A:_R < 1_v >) = A$.

Proposition 2.13. Let A be a semi T-ABSO F. subm. of F. M. X of an R- M. Ḿ. Then $(A:_R x_v)$ is a semi T-ABSO F. ideal of R, for each $x_v \subseteq X - A$.

Proof. Let $r_k^2 b_l \subseteq (A:_R x_v)$ for some F. singletons r_k, b_l of R . Hence $(r_k^2 b_l)x_v \subseteq A$**,** So that $r_k^2(b_l x_v) \subseteq A$**.** Since A is a semi T-ABSO F. subm., then either $r_k b_l x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$, hence either $r_k b_l x_v \subseteq (A:_R x_v)$ or $r_k^2 \subseteq (A:_R X)$. Thus $(A:_R x_v)$ is a semi T-ABSO F. ideal of R.

The following proposition is a characterization of a semi T-ABSO F. subm.

Proposition 2.14. Let A be F. subm. of F. M. X of an R- M. $\acute{M}$. Then A is a semi T-ABSO F. subm. of X iff $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$ or $r_k^2 \subseteq (A:_R X)$ for each F. singletons r_k of R and $x_v \subseteq X$, ∀k,v ∈ L.

Proof. (⟹) Assume that $r_k^2 \nsubseteq (A:_R X)$. To show that $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$.

It is observe that $(A:_R r_k x_v) \subseteq (A:_R r_k^2 x_v)$. Now, let $a_s \subseteq (A:_R r_k^2 x_v)$, hence $r_k^2 a_s x_v \subseteq A$. Since A is semi T-ABSO F. subm. and $r_k^2 \nsubseteq (A:_R X)$, hence $r_k a_s x_v \subseteq A$,
so that $a_s \subseteq (A:_R r_k x_v)$. Then $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$.
(⇐) Let $r_k^2 x_v \subseteq A$, hence $(A:_R r_k^2 x_v) = \lambda_R$ where $\lambda_R(y) = \begin{cases} 1 & if\ y \in R \\ 0 & o.w. \end{cases}$
But $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$ or $r_k^2 \subseteq (A:_R X)$ by hypothesis.
Thus $(A:_R r_k x_v) = \lambda_R$ and then $r_k x_v \subseteq A$. So that either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$.

Definition 2.15. Let $f: \acute{M}_1 \longrightarrow \acute{M}_2$ be a mapping and X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp., then F. kernel of a mapping f denoted by F-ker(f) is F. subm. of X_1 defined by:
F-ker(f)={x_v: $x_v \subseteq X_1$ such that $f(x_v) = 0_1$}, ∀ v∈ L.

Proposition 2.16. Let X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp. Let $f: \acute{M}_1 \longrightarrow \acute{M}_2$ be an epimorphism and A is a semi T-ABSO F. subm. of X_1 such that F- ker $f \subseteq A$. Then $f(A)$ is semi T-ABSO F. subm. of X_2.
Proof. Let $r_k^2 y_h \subseteq f(A)$ for F. singletons r_k of R and $y_h \subseteq X_2$. Since f is onto, so $y_h = f(x_v)$ for some F. singleton $x_v \subseteq X_1$, then $r_k^2 f(x_v) = f(a_s)$ for F. singleton $a_s \subseteq A$. Then $r_k^2 x_v - a_s \subseteq F - \ker f \subseteq A$, thus $r_k^2 x_v \subseteq A$. But A is a semi T-ABSO F. subm., hence $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X_1)$. If $r_k x_v \subseteq A$ then $r_k f(x_v) \subseteq f(a_s)$, hence $r_k y_h \subseteq f(A)$. If $r_k^2 \subseteq (A:_R X_1)$, then $r_k^2 X_1 \subseteq A$, hence $r_k^2 f(X_1) \subseteq f(A)$, thus $r_k^2 \subseteq (f(A):_R f(X_1))$. But $f(X_1) = X_2$ since f is onto, hence $r_k^2 \subseteq (f(A):_R X_2)$.

Remark 2.17. The condition f is an epimorphism in above proposition can't dropped, as can be proved by the following example:
Let X_1:Z→L such that $X_1(y)=\begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$
Let X_2:Z→L such that $X_2(y)=\begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$
It is obvious that X_1, X_2 are F. M. of Z- M. Z.
Let $f: X_1 \longrightarrow X_2$ be F. homomorphism if $f: Z \longrightarrow Z$ with $f(n) = 9n$ be homomorphism but not epimorphism, ∀ $n \in Z$
Let A: Z→L such that $A(y) =\begin{cases} v & if\ y \in 4Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$
It is obvious that A is F. subm. of X_1.
Now, $A_v = 4Z$, $(X_1)_v = Z$ and $(X_2)_v = Z$. $A_v = 4Z$ is a semi T-ABSO subm., but $f(4Z) = 36Z$ is not semi T-ABSO since $2^2.9$∈36Z ,but 2^2∉36Z and 2. 9∉36Z. So that A isa semi T-ABSO F. subm., but $f(A)$ is not semi T-ABSO F. subm.

Proposition 2.18. Let X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp. Let $f: \acute{M}_1 \longrightarrow \acute{M}_2$ be an epimorphism, B is a semi T-ABSO F. subm. of X_2 . Then $f^{-1}(B)$ is a semi T-ABSO F. subm. of X_1.
Proof. Let $r_k^2 x_v \subseteq f^{-1}(B)$ for F. singletons r_k of R and $x_v \subseteq X_1$, hence $f(r_k^2 x_v) \subseteq B$ so $r_k^2 f(x_v) \subseteq B$. Since B is semi T-ABSO F. subm., then either $r_k f(x_v) \subseteq B$ or $r_k^2 \subseteq (B:_R X_2)$, so that $r_k x_v \subseteq f^{-1}(B)$ or $r_k^2 \subseteq (B:_R X_2)$.
If $r_k^2 \subseteq (B:_R X_2)$, then $r_k^2 X_2 \subseteq B$, hence $r_k^2 f(X_1) \subseteq B$. So that $r_k^2 X_1 \subseteq f^{-1}(B)$. Then $r_k^2 \subseteq (f^{-1}(B):_R X_1)$. So that either $r_k x_v \subseteq f^{-1}(B)$ or $r_k^2 \subseteq (f^{-1}(B):_R X_1)$.

Recall that "A F. ideal K of a ring R is called a principle F. ideal if there exists $x_v \subseteq K$ such that $K = \langle x_v \rangle$. For each $a_s \subseteq K$, there exists F. singleton b_l of R such that $a_s = b_l x_v$ where v,s,l ∈ L, that is $K = \langle x_v \rangle = \{a_s \subseteq K: a_s = b_l x_v$ for some F. singleton b_l of R}, [10]".

Proposition 2.19. Let R be a principle F. ideal ring (P. F.I.R) and X be F. M. of an R- M. Ḿ. Let A be a proper F. subm. of X and Ĥ be F. ideal of R . Then A is a semi T-ABSO F. subm. of X iff $\hat{H}^2 x_v \subseteq A$ implies$\hat{H} x_v \subseteq A$ or $\hat{H}^2 \subseteq (A:_R X)$ for any F. ideal Ĥ of R and F. singleton $x_v \subseteq X$.
Proof. (⇒) Suppose that Ĥ be F. ideal of R and F. singleton $x_v \subseteq X$. Since R is P. F.I.R , hence $\hat{H} =< r_k >$ for some F. singleton r_k of R. If $\hat{H}^2 x_v \subseteq A$ then $< r_k >^2 x_v \subseteq A$, thus $r_k^2 x_v \subseteq A$, then either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. Hence $\hat{H} x_v \subseteq A$ or $\hat{H}^2 \subseteq (A:_R X)$
(⇐) It is obvious.

Recall that "Let A and B be two F. subms of F. M. X. If X=A+B and A∩B=0_1, then X is called F. internal direct sum of A and B and denoted by A⊕B. Define by:
(A⊕B)(a,b)=min{A(a), B(b) for all (a,b) ∈ $M_1 \oplus M_2$}
Moreover, A and B are called direct summand of X, [6]".

Proposition 2.20. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1 \oplus \acute{M}_2$ where X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp.. Let A, B be proper F. subms of X_1, X_2 resp., then
1- A is semi T-ABSO F. subm. in X_1 iff $A \oplus X_2$ is semi T-ABSO F. subm. in $X_1 \oplus X_2 = X$.
2- B is semi T-ABSO F. subm. in X_2 iff $X_1 \oplus B$ is semi T-ABSO F. subm. in $X_1 \oplus X_2 = X$.
Proof. (1) (⇒) Let $r_k^2(x_v, y_h) \subseteq A \oplus X_2$ for F. singletons r_k of R and $(x_v, y_h) \subseteq X$. Hence $r_k^2 x_v \subseteq A$ and $r_k^2 y_h \subseteq X_2$. Since A is semi T-ABSO F. subm. in X_1, then either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X_1)$. So that $r_k(x_v, y_h) \subseteq A \oplus X_2$ or $r_k^2 \subseteq (A \oplus X_2:_R X_1 \oplus X_2)$. Then $A \oplus X_2$ is semi T-ABSO F. subm. in $X_1 \oplus X_2 = X$.
(⇐) Let $r_k^2 x_v \subseteq A$ for F. singletonsr_k of R and $x_v \subseteq X_1$, hence for any F. singleton $y_h \subseteq X_2$, $r_k(x_v, y_h) \subseteq A \oplus X_2$. Since $A \oplus X_2$ is a semi T-ABSO F. subm.. in X, then either $r_k(x_v, y_h) \subseteq A \oplus X_2$ or $r_k^2 \subseteq (A \oplus X_2:_R X_1 \oplus X_2) = (A:_R X_1)$. So that $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X_1)$. Then A isa semi T-ABSO F. subm. in X_1.
(2) The proof by the same method in (1).

Proposition 2.21. Let X_1, X_2 be F. M. of an R M. $\acute{M}_1, \acute{M}_2$ resp. and $X = X_1 \oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1 \oplus \acute{M}_2$ such that $F - annX_1 \oplus F - annX_2 = \lambda_R$ where $\lambda_R(y) = 1$, ∀y∈ R. Let A be a semi T-ABSO F. subm. of X, then either
1- $A = A_1 \oplus X_2$ and A_1 is a semi T-ABSO F. subm. in X_1 or
2- $A = X_1 \oplus A_2$ and A_2 is a semi T-ABSO F. subm. in X_2 or
3- $A = A_1 \oplus A_2$ and A_1 is a semi T-ABSO F. subm. in X_1 and A_2 is a semi T-ABSO F. subm. in X_2.
Proof. Since $f - annX_1 \oplus f - annX_2 = \lambda_R$ where $\lambda_R(y) = 1$, ∀y∈ R , then by [5], $A = A_1 \oplus A_2$ for some F. subm. A_1 of X_1 and A_2 of X_2 . Then we have:
(1) $A_1 < X_1$ and $A_2 = X_2$.
(2) $A_1 = X_1$ and $A_2 < X_2$.
(3) $A_1 < X_1$ and $A_2 < X_2$.
Case(1) and case(2), we get $A = A_1 \oplus X_2$ or $A = X_1 \oplus A_2$. Then A_1 is semi T-ABSO F. subm. in X_1 or A_2 is semi T-ABSO F. subm. in X_2 by Proposition (2.20).
Case(3): Suppose that $r_k^2 x_v \subseteq A$ for F. singletons r_k of R and $x_v \subseteq X_1$. Hence $r_k^2(x_v, 0_1) \subseteq A_1 \oplus A_2 = A$. But A be a semi T-ABSO F. subm. of X, then either $r_k(x_v, 0_1) \subseteq A$ or $r_k^2 \subseteq (A:_R X) \subseteq (A_1:_R X_1)$ implies that $r_k x_v \subseteq A_1$ or $r_k^2 \subseteq (A_1:_R X_1)$. Then A_1 is a semi T-ABSO F. subm. in X_1.
By the same method we get A_2 is a semi T-ABSO F. subm.. in X_2

"**Definition 2.22.** A F. M. X of an R- M. M is called a duo F. M. if for each F. subm. A of X, A is fully invariat, [5]".

Note:If $X = X_1 \oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1 \oplus \acute{M}_2$ is a duo F. M. or a distributive F. M. see[9], we can have the same inference of Proposition (2.21).

Proposition 2.23. Let X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp. and A_1, A_2 are semi T-ABSO F. subms of X_1, X_2 resp. such that $(A_1 :_R X_1) = (A_2 :_R X_2)$. Then $A = A_1 \oplus A_2$ is a semi T-ABSO F. subm. of $X = X_1 \oplus X_2$.

Proof. Let $r_k^2(x_v, y_h) \subseteq A_1 \oplus A_2$, so that $r_k^2 x_v \subseteq A_1$ and $r_k^2 y_h \subseteq A_2$. Since A_1, A_2 are semi T-ABSO F. subms, then $r_k x_v \subseteq A_1$ or $r_k^2 \subseteq (A_1 :_R X_1)$ and $r_k y_h \subseteq A_2$ or $r_k^2 \subseteq (A_2 :_R X_2) = (A_1 :_R X_1)$, hence $r_k x_v \subseteq A_1$ and $r_k y_h \subseteq A_2$ or $r_k^2 \subseteq (A_1 :_R X_1)$.
Then $r_k(x_v, y_h) \subseteq A_1 \oplus A_2$ or $r_k^2 \subseteq (A :_R X)$. Thus A is a semi T-ABSO F. subm. of X.

3. Semi T-ABSO F. M.

In this section we present the concept of semi T-ABSO F. M. Some of properties and relationships with other classes of F. M. are illustrated.

First, we give the following definition.

Definition 3.1. A F. M. X of an R- M. $\acute{M}$ is called T-ABSO F. M. if the zero F. subm. (0_1) is T-ABSO F.; that is if for each F. singleton a_s, b_l of R and $x_v \subseteq X$, ∀s,l,v ∈ L, such that $a_s b_l x_v = 0_1$ implies $a_s x_v = 0_1$ or $b_l x_v = 0_1$ or $a_s b_l \subseteq F - annX$.

Now, we present the concept of a semi T-ABSO F. M. as follows:

Definition 3.2. Let X be F. M. of an R- M. $\acute{M}$, X is called a semi T-ABSO F. M. if 0_1 is a semi T-ABSO F. subm. of X.

The proposition specificities a semi T-ABSO F. M. in terms of its level M. is given:

Proposition 3.3. Let X be F. M. of an R- M. $\acute{M}$. Then X is a semi T-ABSO F. M. iff the level X_v is a semi T-ABSO M., ∀ v∈ L.

Proof. (⟹) Let $a^2 x = 0$ for each a∈ R , $x \in X_v$, ∀ v∈ L, then $(a^2 x)_v \subseteq 0_v \subseteq 0_1$, hence $a_s^2 x_k \subseteq 0_1$ where v=min{s, k} and $(a^2)_s = a_s^2$. But 0_1 is a semi T-ABSO F. subm. by Definition (3.2), then either $a_s x_k \subseteq 0_1$ or $a_s^2 \subseteq (0_1 :_R X) = F - annX$, hence $(ax)_v \subseteq 0_1$ or $(a^2)_v \subseteq F - annX$, implies $ax = 0$ or $a^2 \in annX_v$. Then (0) is a semi T-ABSO subm. of X_v.
(⟸) Let $a_s^2 x_k \subseteq 0_1$ for F. singleton a_s of R and $x_v \subseteq X$, then $(a^2 x)_v \subseteq 0_1$ where v=min{s, k}, hence $0_1(a^2x) \geq v$. If $a^2x \neq 0$, then $0_1(a^2x)=0 \geq v$ which is a contradiction. so that $a^2x=0$. But (0) is a semi T-ABSO subm. of X_v, then either $ax = 0$ or $a^2 \in annX_v$, hence $(ax)_v \subseteq 0_1$ or $(a^2)_v \subseteq F - annX$, so that $a_s x_k \subseteq 0_1$ or $a_s^2 \subseteq F - annX$. Thus 0_1 is semi T-ABSO F. subm. of X.

Remarks and Examples 3.4.
(1) Every semiprime F. M. is a semi T-ABSO F. M., but the converse incorrect, for example:

Let X:Z_{49}→L such that $X(y) = \begin{cases} 1 & if\ y \in Z_{49} \\ 0 & o.w. \end{cases}$

It is obvious that X be F. M. of Z- M. Z_{49}.

$X_v = Z_{49}$ as Z- M. is a semi T-ABSO M. since $7^2.\bar{1} = 0$ implies $7^2 \in (0 :_Z Z_{49}) = 49Z$, but X_v is not semiprime M. since 7. $\bar{1}$≠0 . So that X is a semi T-ABSO F. M., but it is not semiprime F. M. by [12].
(2) Every T-ABSO F. M. is a semi T-ABSO F. M.
(3) Every quasi-prime F. M. is a semi T-ABSO F. M. But the converse incorrect see the example in part(1) where $X_v = Z_{49}$ as Z- M. is semi T-ABSO M., but X_v is not quasi-prime M. since 7.7. $\bar{1}$=0 and 7. $\bar{1}$≠0, So that X is semi T-ABSO F. M., but it is not quasi-prime F. M. by [6].
(4) Every F. subm. of a semi T-ABSO F. M. is a semi T-ABSO F. M.

Proposition 3.5. Let X be F. M. of an R- M. $\acute{M}$. If X is a semi T-ABSO F. M., then $F - ann_R X$ is semi T-ABSO F. ideal.

Proof. Since X is semi T-ABSO F. M., then 0_1 is semi T-ABSO F. subm. By Proposition (2.8) when A=0_1, we have $(0_1 :_R X) = F - ann_R X$ is a semi T-ABSO F. ideal.

Proposition 3.6. Let X be a multiplication F. M.of an R- M. $\acute{M}$. Then X is a semi T-ABSO F. M. iff $F - ann_R X$ is a semi T-ABSO F. ideal.

Proof. (⟹) By Proposition (3.5)**,** we get the outcome.
(⟸)By Proposition (2.9)**,** we get the outcome.

Corollary 3.7. Let X be a faithful multiplication F. M. of an R- M. $\acute{M}$. Then the following expressitions are equivalent:
1- X is a semi T-ABSO F. M.;
2- R is a semi T-ABSO F. ring.

Proof. (1) Since X is a semi T-ABSO F. M., so that $F - ann_R X$ is semi T-ABSO F. ideal by Proposition (3.6). But $F - ann_R X = 0_1$, hence 0_1 is semi T-ABSO F. ideal.
Then R is semi T-ABSO F. ring.
(2) Since R is a semi T-ABSO F. ring, so that 0_1 is semi T-ABSO F. ideal, but $0_1 = F - ann_R X$ since X is a faithful. Then X is semi T-ABSO F. M. by Proposition (3.6).

Proposition 3.8. Let X be F. M. of an R- M. $\acute{M}$ such that $F - ann_R X$ is a semiprime F. ideal of R . Then **X** is semi T-ABSO F. M. iff X is semiprime F. M.

Proof. (⟹) Let $r_k^2 x_v \subseteq 0_1$ for F. singletons r_k of R and $x_v \subseteq X$. Since X is semi T-ABSO F. M., then $r_k x_v \subseteq 0_1$ or $r_k^2 \subseteq (0_1 :_R X) = F - ann_R X$. Hence $r_k x_v \subseteq 0_1$ or
$r_k \subseteq F - ann_R X$ since $F - ann_R X$ is semiprime F. ideal of R. Thus $r_k x_v \subseteq 0_1$,$\forall x_v \subseteq X$.Then 0_1 is semiprime F. subm.. So that X is semiprime F. M. by [11].
(⟸) It is obvious.

Proposition 3.9. Let X be F. M. of an R- M. $\acute{M}$. If X is a semi T-ABSO F. M., then $F - ann_R A$ is semi T-ABSO F. ideal for each non-constant F. subm. A of X.

Proof. Let A be a non-empty F. subm. of X and $F - ann_R A \neq \lambda_R$ because if $F - ann_R A = \lambda_R$, then $A = 0_1$ which is a contradiction.
Now, suppose that $r_k^2 a_s \subseteq F - ann_R A$ for F. singletons r_k , a_s of R. Hence $r_k^2 a_s A \subseteq 0_1$. Since **X** is semi T-ABSO F. M., then either $r_k a_s A \subseteq 0_1$ or $r_k^2 \subseteq (0_1 :_R X)$ by Proposition (2.7). Hence either $r_k a_s \subseteq F - ann_R A$ or $r_k^2 \subseteq F - ann_R A$ since $F - ann_R X \subseteq F - ann_R A$ by [6]. Thus $F - ann_R A$ is semi T-ABSO F. ideal.

Recall that "A ring R is said to be an integral domain if R has no zero-divisor F. singleton (i.e. if a_v is F. singleton of R $\exists b_l$ is F. singleton of R such that $a_v b_l = 0_1$,∀v, l ∈L, implies $a_v = 0_1$ or $b_l = 0_1$), [16]".

Recall that " A F. subm A of F. M. X is called a divisible F. if for each F. singleton $x_v \subseteq A$ there exists F. singleton $y_h \subseteq A$ and for each r∈ R, r≠0, $x_v = ry_h$ where $(ry)_h = ry_h$, X is called a divisible F. M. if X is F. divisible subm. of itself, [14]".

Proposition 3.10. Let R is an integral domain and X is a non-empty divisible F. M. of an R- M. $\acute{M}$. Then X is semi T-ABSO F. M. iff X is quasi-prime F. M

Proof. (⟹) Let $r_k a_s x_v \subseteq 0_1$ for F. singletons r_k, a_s of R and $x_v \subseteq X$.
If $r_k a_s \subseteq 0_1$, then $r_k \subseteq 0_1$ or $a_s \subseteq 0_1$, so that $r_k x_v \subseteq 0_1$ or $a_s x_v \subseteq 0_1$.
If $r_k a_s \nsubseteq 0_1$, then $r_k \nsubseteq 0_1$ or $a_s \nsubseteq 0_1$ since R is an integral domain.
If $r_k x_v \subseteq 0_1$, then the proof is complete.
If $r_k x_v \nsubseteq 0_1$, $r_k \nsubseteq 0_1$ and X is a divisible F. M., hence $r_k X = X$, then $x_v = r_k y_h$ for F. singleton $y_h \subseteq X$, thus $r_k a_s x_v = r_k a_s r_k y_h = r_k^2 a_s y_h \subseteq 0_1$. But 0_1 is semi T-ABSO F. subm., then either $r_k a_s y_h \subseteq 0_1$ or $r_k^2 \subseteq F - ann_R X$. If $r_k^2 \subseteq F - ann_R X$ then $r_k^2 X \subseteq 0_1$, but $r_k \nsubseteq 0_1$hence$r_k^2 \nsubseteq 0_1$. Then $r_k^2 X = X \subseteq 0_1$ this is a contradiction. Thus $r_k^2 \nsubseteq F - ann_R X$, then $r_k a_s y_h \subseteq 0_1$ so that $a_s x_v \subseteq 0_1$. Thus 0_1 is quasi-prime F. subm.
(⟸) It is obvious.

Corollary 3.11. Let R be an integral domain and X is a non-empty divisible F. M. of an R- M. Ḿ. Then the following expressions are equivalent:
(1) X is a semi T-ABSO F. M.
(2) X is a quasi-prime F. M.
(3) X is a prime F. M.
Proof. (1)⟺(2) It follows by Proposition(3.10).
(2)⟺(3) It follows by [6].
(3)⟺(1) It follows by [11 , 6]and Proposition(3.10).

Proposition 3.12. A F. M. X of an R- M. Ḿ is a semi T-ABSO F. M. iff either $F - ann\, r_k x_v = F - ann\, r_k^2 x_v$ for any F. singletons r_k of R and $x_v \subseteq X$ such that $r_k x_v \nsubseteq 0_1$ or $r_k^2 X \subseteq 0_1$.
Proof. (⟹) Let $a_s \subseteq F - ann\, r_k^2 x_v$, $r_k^2 x_v \nsubseteq 0_1$.
Then $r_k^2 a_s x_v \subseteq 0_1$. But X is a semi T-ABSO F. M. and $r_k^2 \nsubseteq F - ann\, X$, hence $r_k a_s x_v \subseteq 0_1$, so that $a_s \subseteq F - ann\, r_k x_v$. Then $F - ann\, r_k x_v = F - ann\, r_k^2 x_v$.
(⟸) It is obvious.

Proposition 3.13. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $Ḿ = Ḿ_1 \oplus Ḿ_2$. If X is semi T-ABSO F. M., then X_1 and X_2 are semi T-ABSO F. M.
Proof. By Remarks and Examples(3.4) part(4) the outcome hold.

Remark 3.14. The converse of Proposition(3.13) is not true always, for example:
Let X:$Z_2 \oplus Z_{49}$⟶L such that X(x,y)= $\begin{cases} 1 & if\ (x,y) \in Z_2 \oplus Z_{49} \\ 0 & o.w. \end{cases}$
It is obvious that X be F. M. of Z- M. $Z_2 \oplus Z_{49}$.
And $X_1: Z_2 \longrightarrow$ L such that $X_1(x) = \begin{cases} 1 & if\ \ x \in Z_2 \\ 0 & o.w. \end{cases}$
It is obvious that X_1 be F. M. of Z- M. Z_2.
$X_2: Z_{49} \longrightarrow$ L such that $X_2(y) = \begin{cases} 1 & if\ \ y \in Z_{49} \\ 0 & o.w. \end{cases}$
It is obvious that X_2 be F. M. of Z- M. Z_{49}.
Now, $X_v = Z_2 \oplus Z_{49}$ as Z- M. where $(X_1)_v = Z_2$ and $(X_2)_v = Z_{49}$ are semi T-ABSO M., but $X_v = Z_2 \oplus Z_{49}$ is not semi T-ABSO M. since $7^2(\bar{0}, \bar{1}) = (\bar{0}, \bar{0})$, but $7(\bar{0}, \bar{1}) = (\bar{0}, \bar{7}) \neq (\bar{0}, \bar{0})$ and $7^2 \notin annX_v = annZ_2 \cap annZ_{49} = 2Z \cap 49Z = 98Z$. So that X_1 and X_2 are semi T-ABSO F. M. but X is not semi T-ABSO F. M.

Theorem 3.15. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $Ḿ = Ḿ_1 \oplus Ḿ_2$ where X_1 and X_2 be prime F. M. Then $X = X_1 \oplus X_2$ is semi T-ABSO F. M.
Proof. Let $r_k^2(x_v, y_h) \subseteq (0_1, 0_1)$ for F. singletons r_k of R and $(x_v, y_h) \subseteq X$. Hence $r_k^2 x_v \subseteq 0_1$ and $r_k^2 y_h \subseteq 0_1$, then $r_k(r_k x_v) \subseteq 0_1$ and $r_k(r_k y_h) \subseteq 0_1$. Since X_1 and X_2 be a prime F. M., then either($r_k x_v \subseteq 0_1$ or $r_k \subseteq F - annX_1$) and ($r_k y_h \subseteq 0_1$ or $r_k \subseteq F - annX_2$)
Then there exist four case:
1) If $r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$, then $r_k(x_v, y_n) \subseteq 0_1$.
2) If $r_k \subseteq F - ann\, X_1$ and $r_k \subseteq F - ann\, X_2$, then $r_k \subseteq F - ann\, X_1 \cap F - annX_2 = F - ann\, X$, but $r_k \subseteq F - ann\, X$ implies $r_k^2 \subseteq F - annX$.
3) If $r_k x_v \subseteq 0_1$ and $r_k \subseteq F - ann\, X_2$, then $r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$, hence $r_k(x_v, y_h) \subseteq 0_1$.
4) If $r_k \subseteq F - ann\, X_1$ and $r_k y_h \subseteq 0_1$, then $r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$, hence $r_k(x_v, y_h) \subseteq 0_1$. Then X is a semi T-ABSO F. M.

Remarks 3.16.
(1) By an application of Theorem(3.15), each of the following F. M. is a semi T-ABSO F. M. of an R- M. Z.
X:$Z_p \oplus Z_q$⟶L , X:$Z_p \oplus Z_p$⟶L , X:$Z_p \oplus$Z⟶L , X:Q⊕Z⟶L , X:Z⊕Z⟶L and X:Q⊕Q⟶L where p, q are two prime numbers.
(2) The condition X_1 and X_2 be prime F. M. can't deleted from Theorem (3.15), see Remarks (3.14) where $X_v = Z_2 \oplus Z_{49}$ as Z- M. , $(X_1)_v = Z_2$ as Z- M. is a prime M. and $(X_2)_v = Z_{49}$ as Z- M. is not prime M. also $X_v = Z_2 \oplus Z_{49}$ is not semi T-ABSO M., then X_1 is prime F. M. , X_2 is not prime F. M. and X is not semi T-ABSO F. M.

Proposition 3.17. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $Ḿ = Ḿ_1 \oplus Ḿ_2$ such that $F - annX_1 = F - annX_2$. Then X is semi T-ABSO F. M. iff X_1 and X_2 are semi T-ABSO F. M.
Proof. (⟸) Let $r_k^2(x_v, y_h) \subseteq (0_1, 0_1)$ for F. singletons r_k of R and $(x_v, y_h) \subseteq X$.
Hence $r_k^2 x_v \subseteq 0_1$ and $r_k^2 y_h \subseteq 0_1$. Since X_1 and X_2 be a semi T-ABSO F. M., then either($r_k x_v \subseteq 0_1$ or $r_k^2 \subseteq F - annX_1$) and ($r_k y_h \subseteq 0_1$ or $r_k^2 \subseteq F - annX_2 = F - annX_1$). Thus ($r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$) or $r_k^2 \subseteq F - annX_1$. Then $r_k(x_v, y_h) \subseteq (0_1, 0_1)$ or $r_k^2 \subseteq F - annX_1 = F - annX_1 \cap F - annX_2 = F - annX$.
So that X is semi T-ABSO F. M.
(⟸) It is obvious.

Remarks 3.18. The condition $F - annX_1 = F - annX_2$ is obligate for Proposition (3.17), so we can't dropped it, we see the following example:
Let X:$Z_9 \oplus Q$ ⟶L such that X(x,y)= $\begin{cases} 1 & if\ (x,y) \in Z_9 \oplus Q \\ 0 & o.w. \end{cases}$
It is obvious that X be F. M. of Z- M. $Z_9 \oplus Q$.
And $X_1: Z_9 \longrightarrow$ L such that $X_1(x) = \begin{cases} 1 & if\ \ x \in Z_9 \\ 0 & o.w. \end{cases}$
It is clear that X_1 is F. M. of Z- M. Z_9.
$X_2: Q \longrightarrow$ L such that $X_2(y) = \begin{cases} 1 & if\ \ y \in Q \\ 0 & o.w. \end{cases}$
It is obvious that X_2 be F. M. of Q as Z- M.
Now, $X_v = Z_9 \oplus Q$ as Z- M. and $(X_1)_v = Z_9$ as Z- M. ,
$(X_2)_v = Q$ as Z- M., where $X_v = Z_9 \oplus Q$ is not semi T-ABSO M. since $3^2(\bar{1}, \bar{0}) = (\bar{0}, \bar{0})$, but $3(\bar{1}, \bar{0}) \neq (\bar{0}, \bar{0})$ and $3^2 \notin annX_v = ann_Z Z_9 \cap ann_Z Q = 0$, but each of $(X_1)_v = Z_9$ as Z- M. , $(X_2)_v = Q$ as Z- M. is a semi T-ABSO M. and $ann_Z Z_9 = 9Z \neq ann_Z Q = 0$. So that X is not semi T-ABSO F. M., but X_1 and X_2 are semi T-ABSO F. M. and $F - annX_1 \neq F - annX_2$.
.
Proposition 3.19. The following expressions are equivalent for F. M. X of an R- M. Ḿ
(1) X is a semi T-ABSO F. M.
(2) $F - ann_X$Ĥ is a semi T-ABSO F. subm. for each F. ideal Ĥ of R with Ĥ $\nsubseteq F - annX$.

(3) $F - ann_X < a_s >$ is a sem T-ABSO F. subm. for each F. singleton a_s of R with $a_s \nsubseteq F - annX$, ∀s∈ L.
Proof. It follows by Proposition (2.12) with A=0_1.

Now, we give the concept of a comultiplication F. M. as follows:

Definition 3.20. A F. M. X of an R-M. $\acute{M}$ is called a comultiplication F. M. if A=$\mathrm{F - ann_X F - ann_R}$ A for each F. subm. A of X.

Proposition 3.21. If X is a semi T-ABSO comultiplication F. M. of an R- M. Ḿ. Then every proper F. subm. of X is a semi T-ABSO F. subm.

Proof. Let A be a proper F. subm. of X, hence $A = F - ann_X F - ann_R A$. Put $F - ann_R A = \hat{H}$, so that $A = F - ann_X \hat{H}$. But $\hat{H} \nsubseteq F - ann_R X$ since if $\hat{H} \subseteq F - ann_R X$ hence $F - ann_R X = F - ann_R A$ and then $A = X$ which is a contradication.
Then by Proposition (3.18), $A = F - ann_X \hat{H}$ is semi T-ABSO F. subm. Hence every proper F. subm. A of X is semi T-ABSO F

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